QFI QF Model Solutions
Fall 2021

1.

Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

Hirsa & Neftci 2014 Ch. 8, 9, 10 – 126, 132, 150, 162, 170

Commentary on Question:

Candidates performed fairly on this question.

Solution:

(a) Explain why X_t is a normally distributed random variable.

Commentary on Question:

Candidates performed fairly on this part. To receive full credit, candidates needed to note that we have a Riemann sum and the linear combinations of normal variables are normal and normality is preserved in the limit.

We know that X_t is the mean squared limit of a Riemann sum of the form
$\sum_{i=0}^{n} W_{t_i}(t_i - t_{i-1})$
where $t_0, t_1, \dots, t_n$ is a subdivision of $[0, t]$.

Given that linear combinations of normal random variables are normal and that normality is preserved in the limit, we obtain the desired result.

1. **Continued**

Alternative solution:
By Ito's lemma or Ito's product rule we have

$$d(uW_u) = W_u du + u dW_u,$$

which gives

$$X_t = \int_0^t u\, dW_u - tW_t$$

Both terms on the right-hand side are normally distributed, hence the result.

(b) Compute $E[X_t]$ and $Var[X_t]$.

Commentary on Question:
Candidates performed poorly on this part. Most candidates did not work through all the details of the whole derivation.

By Ito's lemma or Ito's product rule we have

$$d(uW_u) = W_u du + u dW_u,$$

which gives

$$tW_t = X_t + \int_0^t u\, dW_u$$

$$X_t = tW_t - \int_0^t u d\, W_u.$$

Hence

$$E[X_t] = E[tW_t] - E\left[\int_0^t u d\, W_u\right] = 0.$$

We also have

$$\begin{aligned} Var(X_t) &= Var(tW_t) + Var\left(\int_0^t u d\, W_u\right) - 2Cov\left(tW_t, \int_0^t u d\, W_u\right) \\ &= t^2 Var(W_t) + Var\left(\int_0^t u d\, W_u\right) - 2tCov\left(W_t, \int_0^t u d\, W_u\right) \end{aligned}$$

Note that

$$Var(W_t) = t$$

From Ito's isometry, we get

$$Var\left(\int_0^t u d\, W_u\right) = \int_0^t u^2 d\, u = \frac{1}{3} t^3$$

1. Continued

In addition from Ito's isometry, we have

$$2Cov\left(W_t, \int_0^t ud\,W_u\right) = Var\left(W_t + \int_0^t ud\,W_u\right) - Var(W_t) - Var\left(\int_0^t ud\,W_u\right)$$
$$= Var\left(\int_0^t d\,W_u + \int_0^t ud\,W_u\right) - t - \frac{1}{3}t^3$$
$$= Var\left(\int_0^t (u+1)d\,W_u\right) - t - \frac{1}{3}t^3 = \int_0^t (u+1)^2 du - t - \frac{1}{3}t^3 = t^2$$

Hence we have

$$Var(X_t) = \frac{1}{3}t^3$$

Alternative method for calculating $2Cov\left(W_t, \int_0^t ud\,W_u\right)$:

Since $W_t = \int_0^t d\,W_u$, from Ito's isometry

$$2Cov\left(W_t, \int_0^t ud\,W_u\right) = 2Cov\left(\int_0^t d\,W_u, \int_0^t ud\,W_u\right)$$
$$= 2\int_0^t udu = t^2$$

Alternative Solution:

Work with X_t directly, which clearly has mean zero, and then evaluate the expectation of its square as a double integral using the known covariance function of a Wiener process.

$E\left[\int_0^t W_u du\right] = 0$

$Var\left(\int_0^t W_u du\right) = E\left[\left(\int_0^t W_u du\right)^2\right] - \left(E\left[\int_0^t W_u du\right]\right)^2$

$Var\left(\int_0^t W_u du\right) = E[(\left(\int_0^t W_s ds\right)(\left(\int_0^t W_u du\right)] - \left(E\left[\int_0^t W_u du\right]\right)^2$

$= E[(\left(\int_{s=0}^{s=t}\int_{u=o}^{u=t} E[W_s W_u] du\, ds\right)$

$E[W_s W_u] = min(s,u)$

$Var\left(\int_0^t W_u du\right) = \int_{s=0}^{s=t}\int_{u=o}^{u=t} \min(s,u)\, du\, ds$

$= \int_{s=0}^{s=t}\int_{u=o}^{u=s} udu + \int_{s=0}^{s=t}\int_{u=s}^{u=t} sdu\, ds$

$= \int_{s=0}^{s=t} \frac{1}{2}s^2 ds + \int_0^t s(t-s)ds$

$= \frac{1}{3}t^3$

1. Continued

Let Y_t be defined as

$$Y_t = \int_0^t \sqrt{|\mathrm{W}_u|}\, dW_u.$$

(c) Compute $Var\left[Y_t\right]$.

Commentary on Question:
Candidates performed fairly on this part. Most candidates were able to obtain partial credit by identifying the need to use Ito isometry.

We know that $E(Y_t) = 0$ because Y_t is an Ito integral for all $0 < t < T$.
Therefore, $Var(Y_t) = E[Y_t^2] = \int_0^t E(|W_u|)\, du$ by Ito isometry.
Now: $E(|W_u|) = \int_{-\infty}^{\infty} |w| \frac{1}{\sqrt{2\pi u}} e^{-\frac{w^2}{2u}}\, dw = 2\int_0^{\infty} w \frac{1}{\sqrt{2\pi u}} e^{-\frac{w^2}{2u}} dw = \sqrt{\frac{2u}{\pi}}$.
Finally: $Var(Y_t) = \int_0^t \sqrt{\frac{2u}{\pi}}\, du = \sqrt{\frac{2}{\pi}} \frac{2}{3} t^{3/2} = \sqrt{\frac{8}{9\pi}}\, t^{3/2}$.

2. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1j) Understand and apply Girsanov's theorem in changing measures.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014, Chapter 6

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014, Chapter 2 and Chapter 4

Commentary on Question:

Commentary is listed underneath each question component.

Solution:

(a) Show that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures on the probability space implied by the price process A_t.

Commentary on Question:

This question tests the understanding of the definition of equivalent probability measures. Candidates performed as expected on this part.

From the two trees, we see that the sample space and the event space are the same for the probability measures $\mathbb{P}$ and $\mathbb{Q}$. Additionally, $\mathbb{P}(A_t) = 0 \Leftrightarrow \mathbb{Q}(A_t) = 0$ for every event A_t in the event space.

Note: The following statements are equivalent:

- $\mathbb{P}(A_t) = 0 \Leftrightarrow \mathbb{Q}(A_t) = 0$ *for every event* A_t *in the event space.*
- *If an event cannot occur under the* $\mathbb{P}$ *measure, then it also cannot occur under the* $\mathbb{Q}$ *measure, and vice versa.*
- $\mathbb{P}$ *and* $\mathbb{Q}$ *are absolutely continuous with respect to each other.*
- $\mathbb{P} \ll \mathbb{Q}$ *and* $\mathbb{Q} \ll \mathbb{P}$.

2. Continued

(b) Determine if the price process A_t is a:

(i) $\mathbb{Q}$-martingale.

(ii) $\mathbb{P}$-martingale.

Commentary on Question:
Candidates performed well on this part. Most candidates reached the correct conclusions, although many candidates did not check all the conditional expectations under the $\mathbb{Q}$ measure.

Part (i)
A discrete process $X = \{X_n : n = 0,1,2,\dots\}$ is a martingale relative to $(\Omega, \mathcal{F}, \mathbb{P})$ if for all n:

(a) $E(X_{n+1}|\mathcal{F}_n) = X_n$;
(b) $E(|X_n|) < \infty$;
(c) X_n is $\mathcal{F}_n$-adapted.

The last two conditions can be seen trivially from the tree. To check the first condition, with $r = 0$, we can calculate:

$$E^{\mathbb{Q}}(A_2|A_1 = 120) = \frac{3}{5} \times 144 + \frac{2}{5} \times 84 = 120 = A_1$$

$$E^{\mathbb{Q}}(A_2|A_1 = 60) = \frac{4}{7} \times 84 + \frac{3}{7} \times 28 = 60 = A_1$$

$$E^{\mathbb{Q}}(A_1|A_0 = 100) = \frac{2}{3} \times 120 + \frac{1}{3} \times 60 = 100 = A_0$$

Hence, the process A_t is a $\mathbb{Q}$-martingale.

Part (ii)
$E^{\mathbb{P}}(A_2|A_1 = 120) = \frac{1}{2} \times 144 + \frac{1}{2} \times 84 = 114 \neq 120$. Under $\mathbb{P}$, the process A_t violates the martingale property $E(X_{n+1}|\mathcal{F}_n) = X_n$, so it is not a $\mathbb{P}$-martingale.

(c) Calculate the values of the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ for all paths through the tree, (i.e. up-up, up-down, down-up, down-down nodes).

Commentary on Question:
Candidates performed below expectation on this part. Many candidates did not write down the derivatives for $t = 0$ and $t = 1$. Some candidates used a wrong formula for $t = 2$.

2. Continued

The Radon-Nikodym derivative on the finite sample space $\omega \in \Omega$ is a random variable defined as $\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)}$. The sample paths of the tree are ω. Therefore, at each node of the tree, we can calculate the following:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(stationary;\ t = 0) = 1$$
$$\frac{d\mathbb{Q}}{d\mathbb{P}}(up;\ t = 1) = \frac{2/3}{1/2} = \frac{4}{3}$$
$$\frac{d\mathbb{Q}}{d\mathbb{P}}(down;\ t = 1) = \frac{1/3}{1/2} = \frac{2}{3}$$
$$\frac{d\mathbb{Q}}{d\mathbb{P}}(up, up;\ t = 2) = \frac{2/3 \times 3/5}{1/2 \times 1/2} = \frac{8}{5}$$
$$\frac{d\mathbb{Q}}{d\mathbb{P}}(up, down;\ t = 2) = \frac{2/3 \times 2/5}{1/2 \times 1/2} = \frac{16}{15}$$
$$\frac{d\mathbb{Q}}{d\mathbb{P}}(down, up;\ t = 2) = \frac{1/3 \times 4/7}{1/2 \times 1/2} = \frac{16}{21}$$
$$\frac{d\mathbb{Q}}{d\mathbb{P}}(down, down;\ t = 2) = \frac{1/3 \times 3/7}{1/2 \times 1/2} = \frac{4}{7}$$

(d) Evaluate the process $\xi_t = E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right)$ at time $t = 1$ for both up and down nodes where $\mathcal{F}_t$ is the filtration history up to time *t*.

Commentary on Question:
Candidates performed as expected on this part. Partial credits were awarded if candidates wrote down the correct formula but used incorrect results from the last part.

$$E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|A_1 = 120\right) = \frac{1}{2} \times \frac{8}{5} + \frac{1}{2} \times \frac{16}{15} = \frac{4}{3}$$
$$E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|A_1 = 60\right) = \frac{1}{2} \times \frac{16}{21} + \frac{1}{2} \times \frac{4}{7} = \frac{2}{3}$$

2. Continued

(e) Show numerically that $E^{\mathbb{Q}}\left[X\right]=E^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X\right]$ at time 0 by using the results in part (d).

Commentary on Question:
Candidates performed as expected on this part. Most candidates were able to calculate $E^{\mathbb{Q}}[X]$. *Some candidates did not have the incorrect formulae for* $E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}X\right)$.

$$E^{\mathbb{Q}}(X)=\frac{2}{3}*\left(\frac{3}{5}*20+\frac{2}{5}*20\right)+\frac{1}{3}*\left(\frac{4}{7}*20+\frac{3}{7}*0\right)=20*\frac{6}{7}=17.1429$$

$$\begin{aligned}E^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}X\right)=&\ \frac{1}{2}\times\frac{1}{2}\times\frac{d\mathbb{Q}}{d\mathbb{P}}(up,up;\ t=2)\times 20+\\ &\ \frac{1}{2}\times\frac{1}{2}\times\frac{d\mathbb{Q}}{d\mathbb{P}}(up,down;\ t=2)\times 20+\\ &\ \frac{1}{2}\times\frac{1}{2}\times\frac{d\mathbb{Q}}{d\mathbb{P}}(down,up;\ t=2)\times 20+\\ &\ \frac{1}{2}\times\frac{1}{2}\times\frac{d\mathbb{Q}}{d\mathbb{P}}(down,down;\ t=2)\times 0\end{aligned}$$

$$=\frac{1}{2}\times\frac{1}{2}\times 20\times\left(\frac{d\mathbb{Q}}{d\mathbb{P}}(up,up;\ 2)+\frac{d\mathbb{Q}}{d\mathbb{P}}(up,down;\ 2)+\frac{d\mathbb{Q}}{d\mathbb{P}}(down,up;\ 2)\right)$$

$$=\frac{1}{2}\times\frac{1}{2}\times 20\times\left(\frac{8}{5}+\frac{16}{15}+\frac{16}{21}\right)$$

$$=17.1429=E^{\mathbb{Q}}(X)$$

3. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:
Neftci, An Introduction to the Mathematics of Financial Derivatives, Ch. 2, 6, 8, 13.

Commentary on Question:
This question tests candidates' understanding of Ito Lemma and martingales. Candidates did well overall in this question. Several candidates were able to come up with alternative solutions in some parts.

Solution:
(a)

(i) Determine the stochastic differential equation satisfied by the discounted price process $S_t^d = B_t^{-1} S_t$.

(ii) Explain why $\pi_T^d(X) = \pi_t^d(X) + \int_t^T \alpha_u \sigma S_u^d dW_u^{\mathbb{Q}}$.

(iii) Show that the discounted derivative prices $\pi_t^d(X), t < T$ form a $\mathbb{Q}$-martingale using part (a) (ii).

Commentary on Question:
Candidates did well in part (i) and (iii). A common oversight was not using Ito Lemma in part (ii).

Part (i)
$S_t^d = B_t^{-1} S_t = e^{-rt} S_t$.
By Ito's lemma:
$dS_t^d = -re^{-rt} S_t dt + re^{-rt} S_t dt + \sigma e^{-rt} S_t dW_t^Q = \sigma S_t^d dW_t^Q$.

3. Continued

Part (ii)

Self-financing means:

$\pi_u(X) = \alpha_u S_u + \beta_u B_u$ implies $d\pi_u(X) = \alpha_u dS_u + \beta_u dB_u$.

Using Ito's Lemma,

$d\pi_u^d(X) = d\left(e^{-ru}\pi_u(X)\right) = -re^{-ru}\pi_u(X)du + e^{-ru}d\pi_u(X) = -re^{-ru}(\alpha_u S_u + \beta_u B_u)du + e^{-ru}\alpha_u dS_u + e^{-ru}\beta_u dB_u = \alpha_u \,\sigma S_u^d dW_u^Q = \alpha_u \, dS_u^d$

Integrate the above equation from t to T

$\pi_T^d(X) - \pi_t^d(X) = \int_t^T \alpha_u \, dS_u^d$

$X^d - \pi_t^d(X) = \int_t^T \alpha_u \, dS_u^d$ Replicating portfolio

Part (iii)

If s < t then

Using part a(ii)

$\pi_t^d(X) = \pi_s^d(X) + \int_s^t \alpha_u dS_u^d$

Since conditional expectation of the stochastic integral = 0

⇨ $E[\pi_t^d(X)] = \pi_s^d(X)$

So this is a martingale.

(b) Prove that $C_t(K,T) - P_t(K,T) = S_t - Ke^{-r(T-t)}, t < T$.

Commentary on Question:
Candidates did well in this part. Alternate solutions were accepted for full credit as long as they didn't assume the Put-Call parity formula as given.

$C_T - P_T = max(S_T - K, 0) - max(K - S_T, 0)$

$= S_T - K$

$\pi_t(X) = e^{-t(T-t)}E_Q[X|F_t]$ ------ Eq. 1

Apply Eq. 1

$C_t - P_t = e^{-r(T-t)}E[S_T|F_t] - K\, e^{-r(T-t)}$

$C_t - P_t = S_t - K\, e^{-r(T-t)}$

Since $e^{-r(T-t)}E[S_T|F_t] = S_t$ using part (a) (i)

$C_t = C_t(K,T)\ and\ P_t = P_t(K,T)$

(c) Show that $\pi_t^d(V) = P_t(K,T) + C_t\left(Ke^{-r(T-T_c)}, T_c\right), t < T_c$.

Commentary on Question:
Candidates were able to start this part successfully, but most were not able to connect risk-neutral expectation to earn full credit.

3. Continued

Use the put-call parity from part (c),
$max(P_{T_c}, C_{T_c}) = max(P_{T_c}, P_{T_c} + S_{T_c} - Ke^{-r(T-T_c)})$
$max(P_{T_c}, C_{T_c}) = P_{T_c} + max\ (S_{T_c} - Ke^{-r(T-T_c)}, 0)$

Therefore:
$e^{-r(T_c-t)} E[max(P_{T_c}, C_{T_c})|F_t] = e^{-r(T_c-t)} E[P_{T_c} + max(S_{T_c} - Ke^{-r(T-T_c)}, 0)|F_t]$

$= e^{-r(T_c-t)} E[P_{T_c}|F_t] + e^{-r(T_c-t)} E[max(S_{T_c} - Ke^{-r(T-T_c)}, 0)|F_t]$
And $e^{-r(T_c-t)} E[P_{T_c}|F_t] = P_t(K,T)$ using part (b)

$e^{-r(T_c-t)} E[max(S_{T_c} - Ke^{-r(T-T_c)}, 0)\ |F_t] = C_t(Ke^{-r(T-T_c)}, T_c)$

$\pi_t^d(V)$
$= P_t(K,T) + C_t(Ke^{-r(T-T_c)}, T_c)$.

4. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

Neftci, An Introduction to the Mathematics of Financial Derivatives, Ch. 6

Chin, Problems and Solutions in Mathematical Finance, Ch. 1

Commentary on Question:

This question tests candidates' understanding of fundamental stochastic Calculus concepts, such as measurable random variables with respect to an information set and conditional expectation, as well as computational skills involving martingales.

Overall, candidates performed poorly on all four parts of this question.

Solution:

(a) Determine whether X is adapted to the filtration $\{\mathcal{F}_k\}_{1\le k\le n}$.

Commentary on Question:
To receive full credit, candidates need to consider the definition of adapted process, the definition of X (non) measurable, and provide justification.

Following Definition 1.6 from Chin, we have to establish whether X_k is $\mathcal{F}_k$-measurable for all $1 \le k \le n$.

We will show that X_k is not always $\mathcal{F}_k$-measurable by means of a counterexample. For simplicity, take $n = k = 2$. Then

$$\mathcal{F}_2 = \{\emptyset, \{(u,u)\}, \{(u,d)\}, \{(u,u),(u,d)\}, \{(d,u),(d,d)\},\\ \{(u,d),(d,u),(d,d)\}, \{(u,u),(d,u),(d,d)\}, \Omega\}.$$

But ${X_2}^{-1}(0) = \{(u,d),(d,u)\} \notin \mathcal{F}_2$, therefore X_2 is not $\mathcal{F}_2$-measurable. Alternatively, one can show that ${X_2}^{-1}(-2) \notin \mathcal{F}_2$.

4. Continued

(b) Verify that $E^{\mathbb{P}}\left(E^{\mathbb{P}}\left(X_2 \middle| \mathcal{F}_2\right)\right) = E^{\mathbb{P}}\left(X_2\right)$ by direct computation.

Commentary on Question:
To receive full credit, candidates need to describe the information set explicitly and compute the probability distribution of the conditional expectation.

We have $\Omega = \{u,d\}^2 = \{(u,u),(u,d),(d,u),(d,d)\}$. The σ-algebra $\mathcal{F}_2$ has already been described in part (a).

By Definition 1.7 from Chin, we know $Y_2 = E^{\mathbb{P}}(X_2|\mathcal{F}_2)$ is $\mathcal{F}_2$-measurable. Compute Y_2 on $(u,u),(u,d)$ and $\{(d,u),(d,d)\}$ as follows:

- $Y_2(u,u) = \frac{1}{\mathbb{P}(u,u)} X_2(u,u)\mathbb{P}(u,u) = \frac{1}{p^2} 2p^2 = 2$
- $Y_2(u,d) = \frac{1}{\mathbb{P}(u,d)} X_2(u,d)\mathbb{P}(u,d) = \frac{1}{p-p^2} 0\,(p-p^2) = 0$
- $Y_2(\{(d,u),(d,d)\})$
  $= \frac{1}{\mathbb{P}(\{(d,u),(d,d)\})}[X_2(d,u)\mathbb{P}(d,u) + X_2(d,d)\mathbb{P}(d,d)] = \frac{1}{1-p}[0(p-p^2) - 2(1-p)^2]$
  $= -2(1-p)$.

Therefore
$E^{\mathbb{P}}(Y_2) = 2p^2 - 2(1-p)(1-p) = 4p - 2$.
Clearly,
$E^{\mathbb{P}}(X_2) = 2p^2 - 2(1-p)^2 = 4p - 2$, and they are equal.

(c) Show that $S_k, 1 \le k \le n$, forms a martingale with respect to $(\Omega, I, \mathbb{P})$.

Commentary on Question:
To receive full credit, candidates need to demonstrate an understanding of the martingale property, boundedness, and adapted processes.

Let's begin by showing that $E^{\mathbb{P}}(S_{k+1}|I_k) = S_k$. Indeed,

$$E^{\mathbb{P}}(S_{k+1}|I_k) = E^{\mathbb{P}}\left(S_0\left(\frac{1}{p}-1\right)^{X_{k+1}} |I_k\right)$$

$$= E^{\mathbb{P}}\left(S_0\left(\frac{1}{p}-1\right)^{X_k + [1_u(\omega_{k+1}) - 1_d(\omega_{k+1})]} |I_k\right)$$

$$= S_0\left(\frac{1}{p}-1\right)^{X_k} E^{\mathbb{P}}\left(\left(\frac{1}{p}-1\right)^{[1_u(\omega_{k+1}) - 1_d(\omega_{k+1})]}\right)$$

$$= S_k\left[\left(\frac{1}{p}-1\right)p + \left(\frac{1}{p}-1\right)^{-1}(1-p)\right]$$

$$= S_k$$

4. **Continued**

It is also clear that $E^{\mathbb{P}}(|S_k|) < \infty$ and that S_k is I_k-measurable for all $1 \le k \le n$. We thus get the desired result.

(d) Find the value of p for which $E^{\mathbb{P}}\left(\sqrt{S_3}\,|I_2\right) = \sqrt{S_2}$.

Commentary on Question:
This part is similar to part (c) and involves calculations with martingales.

We have

$$E^{\mathbb{P}}\left(\sqrt{S_3}|\mathfrak{I}_2\right) = E^{\mathbb{P}}\left(S_0^{0.5}\left(\frac{1}{p}-1\right)^{0.5X_3}|\mathfrak{I}_2\right)$$
$$= \sqrt{S_2}E^{\mathbb{P}}\left(\left(\frac{1}{p}-1\right)^{0.5[1_u(\omega_3)\,-\,1_d(\omega_3)]}\right)$$
$$= \sqrt{S_2}\left[\left(\frac{1}{p}-1\right)^{0.5}p + \left(\frac{1}{p}-1\right)^{-0.5}(1-p)\right]$$
$$= 2\sqrt{S_2 p(1-p)}.$$

Setting $E^{\mathbb{P}}\left(\sqrt{S_3}|\mathfrak{I}_2\right) = \sqrt{S_2}$, we get

$$2\sqrt{p(1-p)} = 1 \Longrightarrow p = \frac{1}{2}.$$

5. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.

Sources:

Fixed income Securities (chapter 4, 5), Veronesi.

Commentary on Question:

In general, the candidates performed well, particularly in identifying the input data, the formula and the calculation of the 2-year swap rate and its value, and the forward swap rate of the 2-year forward swap contract with expiry of 1 year in parts c), d), and e).

However, the results were below expectation in part b) in the determination of an arbitrage opportunity as to the price of the forward 1-year zero-coupon bond, and in a less proportion, in part a) for the definitions of a forward rate agreement, and interest rate swaps.

Other comments will be made under the appropriate section.

Solution:

(a) Describe forward rate agreements, forward contracts, and interest rate swaps.

Commentary on Question:

For the forwards rate agreement, many did refer to an exchange of a fixed rate versus a floating rate. This was not the completed answer, and the candidates should have been more precise and refer more specially to an exchange of the forward rate versus the future spot rate.

Also, for the interest rate swap, we expected the candidates to go further than describing an exchange of fixed cash flows for floating rate cash flows. To obtain more grading points, the candidates should have completed this answer with mentioning the usual reference of the LIBOR floating rate, the swap rate itself, and the value of the contract at initiation.

Forward Rate Agreement: A FRA is a contract between two counterparties to exchange one cash flow in the future, namely, the forward rate in exchange of the future spot rate.

Forward contract: This is a contract between two counterparties in which they agree that at some predetermined date, they will exchange a security, such as a Treasury note, for a cash price that is also predetermined at initiation of the contract.

5. Continued

Interest Rate Swap: A swap is a contract between two counterparties to exchange cash flows in the future. In a fixed-for-floating swap a counterparty pays a fixed coupon while the other pays a rate linked to a floating rate, typically the LIBOR rate. The fixed rate is called the swap rate, and it is set at initiation of the contract so that the value of the swap is zero.

(b) Determine arbitrage strategy based on the above data.

Commentary on Question:
The calculations and the determination of an arbitrage opportunity as to the price of the forward 1-year zero-coupon quoted by the bank bond ($96,08) versus the implied 1-year forward bond price (($95.60) were well done, but there were fewer good results in the determination of the strategy to take advantage of the situation.
In particular, some candidates but not such many, did complete the answer by describing the cash flows at duration 0, 1, and 2. This was giving additional grading points.

The 2-year bond price is exp(-3.5%*2)*100 = 93.239
The 1 year bond price is exp(-2.5%*1)*100 = 97.531
The implied 1 year forward bond price = 93.239/97.531*100 = 95.600
Yes, there is an arbitrage opportunity.

The company should sell a 1 year bond for $93.71(=96.08*.97531) with notional of $96.08 and agree with the bank to sell the forward bond at $96.08. Buy 2 year zero coupon bond at $95.60

At year 0 – receive $93.71 for the 1 year bond;
pay $93.24 for the 2 year bond
net cash proceeds = $93.71 -$93.24 = $0.47 from the portfolio

At year 1 – pay $96.08 from the 1 year bond purchase
receive $96.08 for selling $100 notioanal 1 year zero coupond bond
net cash proceed is 0

At year 2 - pay $100 from the 1 year zero coupon bond
receive $100 for the 2 year zero coupon bond
net cash proceed is 0

5. Continued

(c) Calculate the 2-year swap rate and the value of the swap at time 0.

Commentary on Question:
Usually well answered.

Swap value at initiation is 0.
$Z(t,T) = e^{-r(t,T)(T-t)}$
Z(0,.5) = exp(-2%*0.5) = 0.99005
Z(0,1) = exp(-2.5%*1) = 0.97531
Z(0,1.5) = exp(-3%*1.5) = 0.95600
Z (0,2) = exp (-3.5%*2) = 0.93239

$$c = n * \left(\frac{1 - Z(0,T_M)}{\sum_{j=1}^{M} Z(0,T_j)}\right) = 2 * \frac{1 - 0.93239}{0.99005 + 0.97531 + 0.95600 + 0.93239} = 3.51\%$$

(d) Calculate the value of the 2-year swap in part (c) at time 0.5 (after cash payment).

Commentary on Question:
Usually well answered to the exception that some have not considered the value of for the P_{FR} (T, T).

At $T_{0.5}$
Z(0.5,1) = exp(-1%*.5) = 0.99501
Z(0.5,1.5) = exp(-1.5%*1) = 0.98511
Z (0.5,2) = exp (-2%*1.5) = 0.97045
Value of swap = Value of floating rate bond – Value of fixed rate bond

$$V_{swap}(t; c, T) = P_{FR}(t, T) - P_c(t, T)$$

$$P_{FR}(T_i, T) = 100$$

$$= 100 - \left(\frac{c}{2} * 100 * \sum_{j=t+1}^{M} Z(T_i, T_j) + 100 * Z(T_i, T_M)\right)$$

$$= 100 - \left(\frac{3.51}{2} * 100 * (0.99501+0.98511+0.97045) + 100*0.97045\right) = -2.22$$

(e) Calculate the forward swap rate of the 2-year forward swap contract with expiry of 1 year.

5. Continued

Commentary on Question:
Some have used the values of the Z's instead of the F's.

Z(0,.5) = exp(-2%*0.5) = 0.99005
Z(0,1) = exp(-2.5%*1) = 0.97531
Z(0,1.5) = exp(-3%*1.5) = 0.95600
Z(0,2) = exp(-3.5%*2) = 0.93239
Z(0,2.5) = exp(-3.5%*2.5)=0.91622
Z(0,3) = exp(-4%*3)=0.88692
F(0,1,1.5) = Z(0.1.5)/Z(0,1) = 0.980199
F(0,1,2) = Z(0.2)/Z(0,1) = 0.955997
F(0,1,2.5) = Z(0.2.5)/Z(0,1) = 0.939413
F(0,1,3) = Z(0.3)/Z(0,1) = 0.909373

$$f_2^s(0,T,T^*) = 2 \times \frac{1 - F(0,T,T^*)}{\sum_{j=1}^{m} F(0,T,T_j)}$$

Where

$$F(t,T_1,T_2) = \frac{Z(t,T_2)}{Z(t,T_1)}$$

= 2*(1-0.909373) / (0.980199+0.955997+0.939413+0.909373)
= 4.79%

6. **Learning Objectives:**

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Ch 5-6

QFIQ-121-20: A Guide to Duration, DV01, and Yield Curve Risk Transformations

Commentary on Question:

Candidates did well overall on this question. Successful candidates understand the pros and cons of different hedging instruments and strategies as well as continuously compounded forward yield curve.

Solution:

(a) Define the following terms:

(i) forward rate f (0,2,5)

(ii) par swap rate s_2

Commentary on Question:

Candidates must define the terms instead of just providing the formulas to get full credit.

(i) f(0,2,5) is the forward rate agreed at time 0 for a risk free investment from time 2 to time 5, compounded continuously.

(ii) s2 is the fixed swap rate that makes the present value of a 2-year swap contract equal to zero, where one party agrees to make 1 fixed payment per year for 2 years on an agreed notional amount while the other party commits to make payments linked to a floating rate index.

(b) Calculate the corresponding rates f_0, s_2, s_5 such that the forward curve matches the swap curve.

6. Continued

Commentary on Question:
Many candidates were able to receive partial credit even if the final answers were inaccurate. Successful candidates were able to use the appropriate formulas and calculate the forward discount factors and the swap rates appropriately.

From formula sheet,

$$c = n \times \left(\frac{1 - Z(0, T_M)}{\sum_{j=1}^{M} Z(0, T_j)} \right)$$

For par swap with 1 year maturity, n = 1, M=1

$s_1 = 2\% = \left(\frac{1 - Z(0,1)}{Z(0,1)}\right)$
Z(0,1) = 1/(1+2%) = 0.9804
f(0,0,1) = ln 0.9804 = 1.98%

$$F(t, T_1, T_2) = e^{-f(t,T_1,T_2)(T_2 - T_1)}$$

$Z(0,T_2) = Z(0,T_1) \times F(0,T_1,T_2)$ where $F(0,T_1,T_2)$ is the forward discount factor between time T_1 and time T_2.

Z(0,2) = Z(0,1) x F(0,1,2) where F(0,1,2) is the forward discount factor between time 1 and time 2.

For par swap with 2 year maturity, n = 1, M=2

$s_2 = \left(\frac{1 - Z(0,2)}{Z(0,1) + Z(0,2)}\right)$

$F(0,1,2) = e^{-f(0,1,2)*(2-1)}$

Substitute and solve for

$s_2 = 2.51\%$

Since the forward curve is assumed to be piece-wise constant, recognize that $F(0,2,T) = e^{-f(0,2,5)*(T-2)}$

$s_5 = \left(\frac{1 - Z(0,5)}{Z(0,1) + Z(0,2) + Z(0,3) + Z(0,4) + Z(0,5)}\right)$

$= 3.02\%$

6. Continued

t	f	$F(0,T_1,T_2)$	$Z(0,T_2)$
1	1.98%	0.98039	0.98039
2	2.99%	0.97054	0.95151
3	3.33%	0.96725	0.92035
4	3.33%	0.96725	0.89021
5	3.33%	0.96725	0.86105

(c) Recommend which hedging instrument to use, and justify your recommendation with 2 supporting arguments.

Commentary on Question:
Most candidates did well on this part. Candidates must state a clear recommendation and provide 2 justifications to receive full credit.

Answer 1: Recommend hedging with forward agreement contracts

Reason for selecting forward agreement contracts:
The available maturity of the bond, or the particular instrument may not be the exact instrument to hedge all of the risk. Using a forward rate agreement, a firm could perfectly hedge the risk. Using futures, the firm would retain some residual risk, as the available instruments (the Eurodollar futures, based on the 3month LIBOR) is not perfectly correlated with the interest rate to hedge (which is a 6-month rate).

The cash flows arising from the futures position accrue over time, which implies the need of the firm to take into account the time value of money between the time at which the cash flow is realized and the maturity of the hedge position. This will call for a reduction in the position in futures.

Answer 2: Recommend hedging with futures

Reason for selecting futures:

Because of their standardization, futures are more liquid than forward contracts, meaning that it is easy to get in and out of the position. For the highly traded futures contracts, such as the 10-year U.S. Treasury note futures or the Eurodollar futures, bid/ask spreads are relatively low and going in and out of positions is relatively inexpensive.

6. **Continued**

The existence of a clearinghouse guarantees performance on futures contracts, while the same may not be true for forward contracts. The clearing house hedges itself through the mark-to-market provision: As soon as one trader's account falls below the margin requirements, a margin call is issued and, if the account is not replenished, the position is closed. This mechanism guarantees to some extent that no large credit exposure is mounted for any single trader.

(d) Construct an appropriate option strategy for each of the two objectives below:

(i) Hedge against the decline in interest rates but still have some upward potential profit from an increase in interest rate.

(ii) Hedge against a significant decline in interest rates such that it will retain some risk but spend less on the protection.

Commentary on Question:
Most candidates received some partial credit. Answers not listed below may be acceptable. Successful candidates were able to construct strategies that fulfill the objectives indicated in the question.

(i) Collar strategy:
Purchase a call option on the underlying 6-month T bills with a strike price K. At the same time, sell a put option on the underlying 6-month T bills with a strike price lower than K.

(ii) Deductibles:
Purchase a call option on the underlying 6-month T bills with a higher strike price than K. Therefore, the call option is only exercised when a significant interest rate decline takes place. The cost of insurance is lower for an exchange of less payoff if the bad event happens.

7. Learning Objectives:

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Chapter 17)

Commentary on Question:

Candidates were expected to demonstrate their understanding of the concepts of risk-neutral measure, forward measure, normalization and the proper application. Full points were awarded with an appropriate formulation of the candidates' logic with the correct answer. Candidates performed well for part (a) but performed poorly for part (c).

Solution:

(a) Write down the matrix equation implied by the Fundamental Theorem of Finance.

Commentary on Question:

Candidates are expected to write the correct matrix equation, and most candidates performed well for part (a). Some candidates, however, added unnecessary assets in the matrix equation. Many candidates constructed the matrix equation in one period. Partial points were awarded in both cases.

$$\begin{bmatrix} 1 \\ 0 \\ B_0 \end{bmatrix} = \begin{bmatrix} (1+r_0)(1+r_1^u) & (1+r_0)(1+r_1^u) & (1+r_0)(1+r_1^d) & (1+r_0)(1+r_1^d) \\ F_0 - L_1^u & F_0 - L_1^u & F_0 - L_1^d & F_0 - L_1^d \\ 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} \psi^{uu} \\ \psi^{ud} \\ \psi^{du} \\ \psi^{dd} \end{bmatrix}$$

(b) Calculate the following:

(i) Spot rate r_0 at time t_0

(ii) Spot rates r_1^i at time t_1 for $i = u, d$

7. Continued

Commentary on Question:
Candidates performed well for part (i). Some candidates used continuous compounding, and full points were awarded. Candidates did not perform well for part (ii). When candidates demonstrated the correct logic without the right answer, partial points were awarded.

(i) We know there are two possible states for period $t_0 - t_1$:
Using the matrix, we have

$$(1 + r_0)\psi^u + (1 + r_0)\psi^d = 1$$

We know the one-year zero coupon bond price = 1 / $(1 + r_0)$ = 0.980392

$$r_0 = \frac{1}{0.980392} - 1 = 0.02$$

(ii) $(1 + r_0)(1 + r_1^u)\psi^{ud} = \mathbb{Q}_{ud} = 0.28$

Given ψ^{ud} = 0.267814, we can solve for r_1^u

$$r_1^u = \frac{0.28}{0.267814\ (1 + 0.02)} - 1 = 0.025$$

$$1 = (1 + r_0)(1 + r_1^u)\psi^{uu} + (1 + r_0)(1 + r_1^u)\psi^{ud} + (1 + r_0)\left(1 + r_1^d\right)\psi^{du} + (1 + r_0)(1 + r_1^d)\psi^{dd}$$

$\mathbb{Q}_{uu} + \mathbb{Q}_{ud} + (1 + r_0)\left(1 + r_1^d\right)(\psi^{du} + \psi^{dd}) = 1$ – Equation (1)
$\psi^{uu} + \psi^{dd} + \psi^{du} + \psi^{dd} = B_0 = 0.960742$ – Equation (2)
$\psi^{du} + \psi^{dd} = 0.960742 - 0.401722 - 0.267814 = 0.291206$

$$\left(1 + r_1^d\right) = \frac{1 - 0.42 - 0.28}{(1 + 0.02)(0.291206)}$$

$$r_1^d = 0.01$$

(c) Show that F_0 is not an unbiased estimator of L_1 under classical risk-neutral measure.

Commentary on Question:
Many candidates did not perform very well. Most candidates demonstrated the last line without showing all steps. When the candidate's logic was correct, partial points were awarded.

$$0 = (F_0 - L_1^u)\psi^{uu} + (F_0 - L_1^u)\psi^{ud} + \left(F_0 - L_1^d\right)\psi^{du} + \left(F_0 - L_1^d\right)\psi^{dd}$$

Multiply and divide each term on RHS by corresponding $(1 + r_0)\left(1 + r_1^i\right)$ and relabel using $(1 + r_0)\left(1 + r_1^i\right)\psi^{ij} = \mathbb{Q}_{ij}$

7. Continued

$$0 = \frac{(F_0 - L_1^u)}{(1+r_0)(1+r_1^u)}\mathbb{Q}_{uu} + \frac{(F_0 - L_1^u)}{(1+r_0)(1+r_1^u)}\mathbb{Q}_{ud} + \frac{\left(F_0 - L_1^d\right)}{(1+r_0)\left(1+r_1^d\right)}\mathbb{Q}_{du} + \frac{\left(F_0 - L_1^d\right)}{(1+r_0)\left(1+r_1^d\right)}\mathbb{Q}_{dd}$$

$$F_0\left[\frac{\mathbb{Q}_{uu}}{(1+r_0)(1+r_1^u)} + \frac{\mathbb{Q}_{ud}}{(1+r_0)(1+r_1^u)} + \frac{\mathbb{Q}_{du}}{(1+r_0)\left(1+r_1^d\right)} + \frac{\mathbb{Q}_{dd}}{(1+r_0)\left(1+r_1^d\right)}\right] = \left[\frac{L_1^u\mathbb{Q}_{uu}}{(1+r_0)(1+r_1^u)} + \frac{L_1^u\mathbb{Q}_{ud}}{(1+r_0)(1+r_1^u)} + \frac{L_1^d\mathbb{Q}_{du}}{(1+r_0)\left(1+r_1^d\right)} + \frac{L_1^d\mathbb{Q}_{dd}}{(1+r_0)\left(1+r_1^d\right)}\right]$$

$$F_0\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{(1+r_0)(1+r_1)}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{L_1}{(1+r_0)(1+r_1)}\right]$$

$$F_0 = \frac{1}{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{(1+r_0)(1+r_1)}\right]}\mathbb{E}^{\mathbb{Q}}\left[\frac{L_1}{(1+r_0)(1+r_1)}\right]$$

$$F_0 \neq \mathbb{E}^{\mathbb{Q}}[L_1]$$

(d) Show that F_0 is an unbiased estimator of L_1 under the forward measure with numeraire equals to the two-period zero-coupon bond (Zero-coupon bond maturity at time $T = t_2$).

Commentary on Question:
Candidates performed well if they demonstrated the correct approach in part (c). Some candidates did not show calculations, but correctly referenced part (c), and full points were awarded.

$$\psi^{uu} + \psi^{dd} + \psi^{du} + \psi^{dd} = B_0$$

Dividing by B_0, $\frac{1}{B_0}\psi^{uu} + \frac{1}{B_0}\psi^{dd} + \frac{1}{B_0}\psi^{du} + \frac{1}{B_0}\psi^{dd} = 1$

Using $\pi^{ij} = \frac{1}{B_0}\psi^{ij}$,

$$0 = (F_0 - L_1^u)\psi^{uu} + (F_0 - L_1^u)\psi^{ud} + \left(F_0 - L_1^d\right)\psi^{du} + \left(F_0 - L_1^d\right)\psi^{dd}$$

Multiply and divide each term on RHS by corresponding B_0 and relabel using $\frac{1}{B_0}\psi^{ij} = \pi^{ij}$

$$0 = (F_0 - L_1^u)B_0\pi^{uu} + (F_0 - L_1^u)B_0\pi^{ud} + \left(F_0 - L_1^d\right)B_0\pi^{du} + \left(F_0 - L_1^d\right)B_0\pi^{dd}$$
$$F_0\left(\pi^{uu} + \pi^{ud} + \pi^{du} + \pi^{dd}\right) = L_1^u\pi^{uu} + L_1^u\pi^{ud} + L_1^d\pi^{du} + L_1^d\pi^{dd}$$
$$\pi^{uu} + \pi^{ud} + \pi^{du} + \pi^{dd} = 1$$

$$F_0 = L_1^u\pi^{uu} + L_1^u\pi^{ud} + L_1^d\pi^{du} + L_1^d\pi^{dd} = \mathbb{E}^{\pi}[L_1]$$

8. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3b) Understand and apply various one-factor interest rate models.

(3c) Calibrate a model to observed prices of traded securities.

(3d) Describe the practical issues related to calibration, including yield curve fitting.

(3l) Demonstrate an understanding of the issues and approaches to building models that admit negative interest rates.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Chapter 20)

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010

QFIQ-129-21 Section 3 4 - Negative Interest Rates and Their Technical Consequences_AAE_12 2016

Commentary on Question:

This question generally tests candidates' understanding of the following concepts:

- *Ito's Lemma*
- *Bond pricing*
- *Yield curve*
- *Implications of a zero floor for interest rate under different scenarios*

Most candidates attempted the question and performed as expected.

Solution:

(a) Show that

$$\frac{r_t B_1 - \frac{\partial B_1}{\partial t} - \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2}{\frac{\partial B_1}{\partial r}} = \frac{r_t B_2 - \frac{\partial B_2}{\partial t} - \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2}{\frac{\partial B_2}{\partial r}}$$

8. Continued

Commentary on Question:
Most candidates were able to list out the key points in this question:
- *Condition to eliminate the interest rate risk*
- *Risk free portfolio should earn risk-free rate*

Full marks were awarded to candidates who showed all the derivation.

Using Ito's Lemma:

$$dB_i = \frac{\partial B_i}{\partial t}dt + \frac{\partial B_i}{\partial r}dr_t + \frac{1}{2}\frac{\partial^2 B_i}{\partial r^2}(dr_t)^2$$
$$= \frac{\partial B_i}{\partial r}(a(b-r_t)dt + \ \sigma dW_t) + \frac{\partial B_i}{\partial t}dt + \frac{1}{2}\frac{\partial^2 B_i}{\partial r^2}(a(b-r_t)dt + \sigma dW_t)^2$$
$$= \left(\frac{\partial B_i}{\partial r}a(b-r_t) + \frac{\partial B_i}{\partial t} + \frac{1}{2}\frac{\partial^2 B_i}{\partial r^2}\sigma^2\right)dt + \frac{\partial B_i}{\partial r}\sigma dW_t$$

Plugging to $dP = \eta dB_1 - \theta dB_2$:

$$dP = \eta\left[\left(\frac{\partial B_1}{\partial r}a(b-r_t) + \frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right)dt + \frac{\partial B_1}{\partial r}\sigma dW_t\right]$$
$$-\theta\left[\left(\frac{\partial B_2}{\partial r}a(b-r_t) + \frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)dt + \frac{\partial B_2}{\partial r}\sigma dW_t\right]$$

For interest rate risk to be elimated,

$$\frac{\partial P}{\partial r} = \left(\eta\frac{\partial B_1}{\partial r} - \theta\frac{\partial B_2}{\partial r}\right)a(b-r_t) = 0$$

Choose η and θ such that $\eta\frac{\partial B_1}{\partial r} = \theta\frac{\partial B_2}{\partial r}, i.e., \theta = \eta\frac{\frac{\partial B_1}{\partial r}}{\frac{\partial B_2}{\partial r}}$

$$dP = \left[\eta\left(\frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right) - \theta\left(\frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)\right]\ dt + \sigma(\eta\frac{\partial B_1}{\partial r} - \theta\frac{\partial B_2}{\partial r})dW_t$$

Then dP is riskless, its deterministic return should equal to risk-free rate:$dP = rPdt$.

$$\left[\eta\left(\frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right) - \theta\left(\frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)\right]dt = r(\eta B_1 - \theta B_2)dt$$

Since $\theta = \eta\frac{\frac{\partial B_1}{\partial r}}{\frac{\partial B_2}{\partial r}}$, we have

8. **Continued**

$$\eta r\left(B_1 - \frac{\partial B_1/\partial r}{\partial B_2/\partial r}B_2\right) = \eta\left[\left(\frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right) - \frac{\partial B_1/\partial r}{\partial B_2/\partial r}\left(\frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)\right]$$

$$r\left(B_1 - \frac{\partial B_1/\partial r}{\partial B_2/\partial r}B_2\right) = \left(\frac{\partial B_1}{\partial t} + \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2\right) - \frac{\partial B_1/\partial r}{\partial B_2/\partial r}\left(\frac{\partial B_2}{\partial t} + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)$$

$$rB_1 - \frac{\partial B_1}{\partial t} - \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2 = \frac{\partial B_1/\partial r}{\partial B_2/\partial r}\left(rB_2 - \frac{\partial B_2}{\partial t} - \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2\right)$$

$$\frac{rB_1 - \frac{\partial B_1}{\partial t} - \frac{1}{2}\frac{\partial^2 B_1}{\partial r^2}\sigma^2}{\partial B_1/\partial r} = \frac{rB_2 - \frac{\partial B_2}{\partial t} - \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}\sigma^2}{\partial B_2/\partial r}$$

(b) Show that the price of a default-free discount bond satisfies the following partial differential equation

$$\frac{\partial B}{\partial r}\left[a\left(b - r_t\right) - \sigma\lambda\right] + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2 - r_t B = 0$$

Commentary on Question:
Candidates did well in this question.

From

$$\lambda(r_t, t) = \frac{a(b - r_t) - m(t)}{\sigma}$$

We have $m(t) = a(b - r_t) - \sigma\lambda$.

$$\frac{r_t B - \frac{\partial B}{\partial t} - \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2}{\frac{\partial B}{\partial r}} = a(b - r_t) - \sigma\lambda$$

Therefore, we have PDE:

$$\frac{\partial B}{\partial r}[a(b - r_t) - \sigma\lambda] + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2 - r_t B = 0$$

8. Continued

(c) Describe the key features of this interest rate model.

Commentary on Question:
Full marks were given to candidates who identified 4 key features. Most candidates were able to list at least two key features.

- Interest rate is mean reverting, which means that if they diverge too much from a central value, they tend to revert back to it.
- The model has constant volatility
- The model implies that the statistical distribution of interest rates in the future is normal.
- It gives positive probability to negative nominal interest rate.
- The solution to the fundamental pricing equation under the Vasicek model is known in closed form

(d) Explain how to estimate the interest rate model parameters, using the given data. Identify the estimated parameters that can be used in pricing interest rate derivative.

Commentary on Question:
Most candidates stated the estimation of risk-neutral parameters, not the real-world parameters.
Most candidates did not identify the estimated parameters that can be used in pricing interest rate derivatives.

- The volatility σ can be estimated directly from the time series of interest rate r_t.
- Compute long-run mean of spot rate over the sample period.
- Obtain speed of mean reversion by regressing the changes in interest rate.
- The volatility σ can be used in the bond pricing formula. The estimated long run mean and speed of mean reversion are not relevant for pricing interest rate securities.

(e) Calculate the default-free discount bond price with 30-year maturity with r = 0.1%, 5%, and 10%, respectively.

Commentary on Question:
Most candidates did not perform well in this question. Candidates used the formula from Pietro textbook, and partial marks were awarded.

8. Continued

Using the formula:

$$B = e^{\frac{1}{a}(1-e^{-aT})(R-r)-TR-\frac{\sigma^2}{4a^3}(1-e^{-aT})^2}$$

$$R = b - \frac{\sigma\lambda}{a} - \frac{\sigma^2}{a^2}$$

$a = 0.25$
T = 30
$b = 0.05$
σ = 0.015
λ = -0.1

Spot rate	30-year zero coupon bond
0.1%	0.254079
5%	0.208879
10%	0.171034

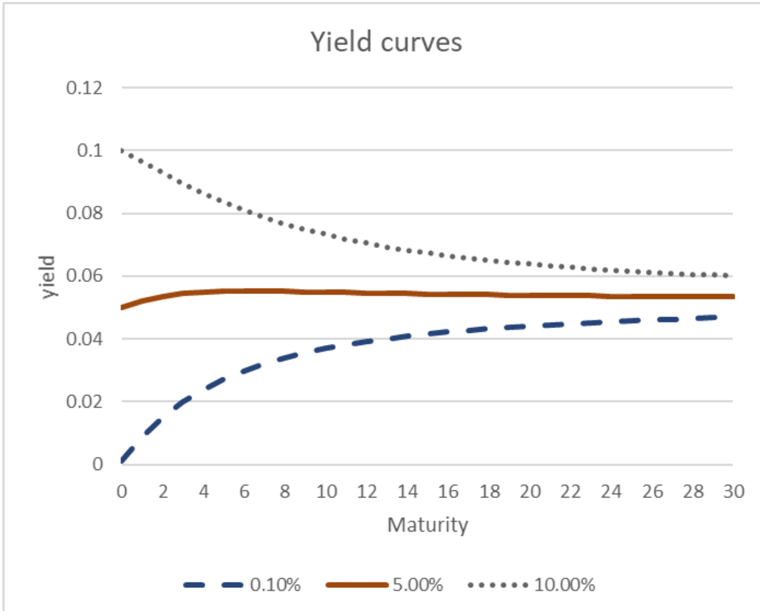
(f) Generate the yield curves for the same set of spot rates in part (e) with different maturities,1 through 30 years.

Commentary on Question:
Candidates performed below expectation for this question. About half of the candidates did not attempt this question. Full marks were awarded for candidates who used the yield formula to solve the question.

$$yield = -\frac{\ln(B)}{T}$$

Spot rate	1	5	10	30 years
0.1%	0.71%	2.343%	3.383%	4.567%
5%	5.045%	5.14%	5.182%	5.220%
10%	9.469%	7.994%	7.018%	5.886%

8. Continued



(g) Analyze the impact of zero floor on

(i) cost of guarantees;

(ii) discount factors and best estimate values;

(iii) martingality of economic scenarios and company's capital position.

Commentary on Question:
This question was from "Section 3 4 - Negative Interest Rates and Their Technical Consequences_AAE_12 2016".

- The cost of financial options and guarantees could be underestimated
- The discount factors used to discount the cash-flow part of the Best Estimate would be artificially high, leading to an underestimation of the Best Estimate.
- The martingality of economic scenarios would be broken, therefore, estimators of the Best Estimate of Technical Provisions would not converge to the correct answer.
- This deterioration in the estimators' convergence could then result in an inflated capital position.

9. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(3b) Understand and apply various one-factor interest rate models.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

(3j) Understand and apply the Heath-Jarrow-Morton approach including the LIBOR Market Model.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010 (Ch15, Ch 19, Ch. 21)

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch 9, 10, 19)

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Show by using Ito's lemma that

$$df\left(t,T\right)=\left[\frac{\partial B\left(t,T\right)}{\partial T}\left(\theta_t-\gamma_t r_t\right)+r_t\frac{\partial^2 B\left(t,T\right)}{\partial t\partial T}-\frac{\partial^2 A\left(t,T\right)}{\partial t\partial T}\right]dt$$

$$+\frac{\partial B\left(t,T\right)}{\partial T}\sqrt{\sigma_t^2+a_t r_t}\,dX_t.$$

9. Continued

Commentary on Question:
Most Candidates did well for this question. Candidates who did not receive full points because they were not able to apply Ito's Lemma successfully.

$$f(t,T) = -\frac{\partial \ln Z(t,T)}{\partial T} = -\frac{\partial A(t,T)}{\partial T} + r(t)\frac{\partial B(t,T)}{\partial T}$$

$$df(t,T) = \left[\frac{\partial B(t,T)}{\partial T}\, dr_t + r(t)\frac{\partial^2 B(t,T)}{\partial t \partial T} - \frac{\partial^2 A(t,T)}{\partial t \partial T}\right\} dt$$

$$= \left[\frac{\partial B(t,T)}{\partial T}\,(\theta_t - \gamma_t r_t) + r(t)\frac{\partial^2 B(t,T)}{\partial t \partial T} - \frac{\partial^2 A(t,T)}{\partial t \partial T}\right\} dt$$
$$+ \frac{\partial B(t,T)}{\partial T}\sqrt{\sigma_t^2 + \alpha_t r_t} dX_t \, .$$

(b) Describe the necessary inputs for pricing derivative securities within the HJM framework.

Commentary on Question:
Most candidates did not provide sufficient inputs to receive full credits. Most candidates identified only volatility of bonds or forward rates or initial term structure of interest rates. Only very few candidates identified no arbitrage or risk neutral.

Solution:

Heath, Jarrow, and Morton (HJM) framework shows that the risk-neutral dynamics of forward rates are fully characterized by the volatility of forward rates.

Given a volatility structure of either bonds or forward rates and the initial term structure of interest rates, nothing else is necessary to price derivative securities by no arbitrage.

(c) Assess whether the Ho-Lee model and the Black and Karasinski model belong to the class of generalized affine models. Justify your answer.

Commentary on Question:
Most Candidates did well for this question. Some candidates did not receive full credit because they failed to provide justifications.

9. Continued

The Ho-Lee model belongs to the class of generalized affine models.

When $\gamma_t = \gamma, \sigma_t = \sigma,\ \alpha_t = 0$, it corresponds to the Ho-Lee Model.

The Black and Karasinski model does not belong to the class of generalized affine models as the drift term $r_t(\theta_t + \sigma^2/2 - \gamma_t \ln r_t)$ under the Black and Karasinski model cannot be written in the form of $\theta_t - \gamma_t r_t$.

(d) Demonstrate that under the Ho-Lee model:

(i) $\sigma_f(t,T) = \sigma$ using part (a).

(ii) $f(t,T) = r_t + f(0,T) - f(0,t) + \sigma^2 t(T-t)$ where r_t is the short rate.

Commentary on Question:
Most candidates did well in part (i) but most candidates did not do well in part (ii). For part (i), many candidates did not correctly identify B(t,T)=T-t. For part(ii), most candidates did not attempt it or failed to solve for df(t,T) based on HJM formula.

(i) By part (b),

$$\sigma_f(t,T) = \frac{\partial B(t,T)}{\partial T}\sqrt{\sigma_t^2 + \alpha_t r_t}$$

Under the Ho-Lee model:
$\alpha_t = 0, B(t,T) = T - t, \sigma_t = \sigma, \frac{\partial B(t,T)}{\partial T} = 1$

As such, $\sigma_f(t,T) = 1 * \sigma = \sigma$

(ii) Under the Ho-Lee model:

$$\sigma_f(t,T)\int_t^T \sigma_f(t,\tau)\, d\tau$$
$$= \sigma \int_t^T \sigma\, d\tau$$
$$= (T-t)\sigma^2$$

$$df(t,T) = \sigma^2(T-t) + \sigma\, dX_t$$

9. Continued

Integrating the above equation with respect to time t,

$$f(t,T) = f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma X_t$$

Setting T=t, gives the short rate since $r_t = f(t,t)$

$$r_t = f(t,t) = f(0,t) + \frac{\sigma^2 t^2}{2} + \sigma X_t$$

Isolating the Brownian term in terms of forward rates and making substitutions gives

$$f(t,T) = f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + f(t,t) - f(0,t) - \frac{\sigma^2 t^2}{2}$$

$$f(t,T) = r_t + f(0,T) - f(0,t) + \sigma^2 t(T - t)$$

(e) Demonstrate that under the Ho-Lee model:

(i) $Var\left[r_{T_0}\right] = T_0\sigma^2$

(ii) $Var\left[\log\left(Z\left(r_{T_0}, T_0; T_B\right)\right)\right] = \sigma^2 \cdot T_0\left(T_B - T_0\right)^2$

where $Z\left(r_{T_0}, T_0; T_B\right)$ is the value at time T_0 of a zero-coupon bond with maturity date T_B with $T_B > T_0$.

Commentary on Question:
Most Candidates did well for this question. Common errors were candidates did not demonstrate sufficient details to show how they arrived at their final solution.

(i) $dr_t = \theta_t dt + \sigma dX_t$

Integrating both sides from 0 to T_o

$$r_{T_o} - r_0 = \int_0^{T_o} \theta_t dt + \int_0^{T_o} \sigma dX_t$$

$Var[r_{T_o}]$ =$[E\{\left(\int_0^{T_o} \sigma dX_t\right)^2]$

9. Continued

By Ito Isometry
$= E[\int_0^{T_O} \sigma^2 dt]$
$=\sigma^2 T_O$

(ii)
$Z(r,t,T) = e^{A(t,T)-B(t,T)r}$
$Var[\log(Z_{rT_O}, T_O; T_B)]$
$= Var[A(T_O;T_B) - B(T_O;T_B)\ x\ r_{T_o}]$
$= B(T_O;T_B)^2\ x\ Var[r_{T_o}]$
$= (T_B - T_O)^2\ x\ \sigma^2 T_O$

(f)

(i) Calculate exercise price K.

(ii) Compute the value at time $t = 0$ of the above European call option on the zero-coupon bond.

Commentary on Question:
In general, most candidates did poorly for this question or did not attempt at all. For part (i), most candidates struggled to identify the correct formula to calculate the bond price using the forward rates function f(0,T). Some candidates tried unsuccessfully to solve for the bond price using the much more difficult method of Z(r,t,T)=e$^{A(t,T)-B(t,T)*r}$. For (ii), most candidates made the mistake when solving for $s_Z\,(T_o, T_B)$ either by forgetting to take the square root or inputting the wrong values.

(i) $Z(0,T) = e^{-\int_0^T f(0,u)\,du}$

$= e^{-\int_0^T (0.02+0.002u)\,du} = e^{-(0.02T+0,001T^2)}$

Z(0,2)= $e^{-(0.02*2+0,001*2^2)} = 0.956953957$

Z(0,10)= $e^{-(0.02*10+0,001*10^2)} = 0.740818221$

$K = \frac{Z(0,10)}{Z(0,2)} = \frac{0.740818221}{0.956953957} = 0.774141969\ for\ ATM\ option$

(ii) $s_Z\,(T_O, T_B)^2 = \sigma^2 * T_O(T_B - T_O)^2$ =0.005^2*2*(10-2)^2=0.0032

$Z(0, r_0; T_B)/KZ(0, r_0 T_o) = 1\ for\ ATM\ option$

$$d_1 = \frac{1}{s_Z\,(T_o, T_B)} \ln\left(\frac{Z(0, r_0; T_B)}{K * Z(0, r_0 T_o)}\right) + \frac{s_Z\,(T_o, T_B)}{2}$$

9. **Continued**

$$d_1 = \frac{1}{\sqrt{0.0032}} \ln(1) + \frac{\sqrt{0.0032}}{2}$$

$= 0.028284271$

$d_2 = d_1 - s_Z\,(T_o, T_B) = 0 + \frac{s_Z\,(T_o, T_B)}{2} - s_Z\,(T_o, T_B) = -d_1$

$N(d_2) =$ 0.488717713

The price of the first call option
$V(r_0) = Z(0, r_0; T_B)N(d_1) - K_I Z(0, r_0; T_O)N(d_2)$
= $Z(0,10)\left(1 - 2\,N(d_2)\right) = 0.016716248$

10. Learning Objectives:

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3b) Understand and apply various one-factor interest rate models.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Ch 14-22

Commentary on Question:

There was a critical typo in part (b) that may have caused confusion for candidates. As part (c) also relies on part (b), it may have also been confusing for candidates. As a result, full credit was given for parts (b) and (c) for all candidates.

Solution:

(a) Calculate the option price for the issued bond at $t = 0$.

Commentary on Question:

Many candidates did not recognize that this was a put option, not a call option. Some candidates calculated the zero-bond prices by calculating A(t,T) and B(t,T) from first principles (rather than taking values from the provided table. Doing so would result in slightly different option pricing values. Candidates were not penalized for doing this.

The option is worth when the bond price is below 95M, which is a put option with strike at 95M. The payoff at t=1 is

$$V = max(95 - 100 \times Z(1,3), 0) = 100 \times max(K - Z(1,3), 0),$$
$$K = 0.95$$

The pricing formula is

$$V(r_0, 0,1) = KZ(0,1)N(-d_2) - Z(0,3)N(-d_1)$$
$$d_1 = \frac{1}{S_Z(1)} ln\left(\frac{Z(0,3)}{KZ(0,1)}\right) + \frac{S_Z(1)}{2}$$
$$d_2 = d_1 - S_Z(1)$$
$$S_Z(1) = B(1,3)\sqrt{\frac{\sigma^2}{2\gamma^*}(1 - e^{-2\gamma^* \times 1})}$$

10. **Continued**

From the zero-bond pricing formula:

$$Z(r,t,T) = e^{A(t,T)-B(t,T)r}$$

We have

$$Z(r_0,0,1) = e^{-0.01-0.80*2\%} = 0.97434$$
$$Z(r_0,0,3) = e^{-0.09-1.62*2\%} = 0.8848$$

$$S_Z(1) = 1.3\sqrt{\frac{2.21\%^2}{2*0.4653}(1-e^{-2*0.4653\times 1})} = 0.023178$$

$$d_1 = \frac{1}{0.023178} ln\left(\frac{0.8848}{0.95\times 0.97434}\right) + \frac{0.023178}{2} = -1.934$$
$$d_2 = d_1 - 0.023178 = -1.958$$

Using given table values

$$V(r_0,0,1) = 0.95*0.97434*N(-d_2) - 0.8848*N(-d_1) =$$
$$= 0.95*0.97434*0.97488 - 0.8848*0.97344 = 0.041$$

Therefore the option value is 100*0.041 = 4.10 M

(b) Specify an appropriate replicating portfolio.

Commentary on Question:
The typo in the question was to hedge the 3-year bond using the 3-year bond. The intention was to hedge the 3-year bond using the 5-year bond.

The replicating portfolio for the company 3-year bond consists of the 5-year bond and cash.

$$Z(r_0,0,5) = e^{-0.19-1.94*2\%} = 0.7955$$
$$P_t = \Delta Z_2(0,5) + C$$

Where the hedge ratio

$$\Delta = \frac{B(0,3)Z(r,0,3)}{B(0,5)Z(r,0,5)} = \frac{1.62*.8848}{1.94*0.7955} = 0.9288$$

5-year bond position: 0.9288*79.55
Cash position: $C = 88.48 - 0.9288*79.55 = 14.60$ M
Therefore, the replicating portfolio has value of $Z(r,0,3)$.
$P_t = \Delta Z_2(0,5) + C = 0.9288*79.55 + 14.60 = 88.48$

10. Continued

(c) Compute the position of $Z(0,5)$ and cash position of the hedge portfolio when the underlying bond is replicated as in part (b).

Commentary on Question:
This part was not graded due to potential confusion arising from part (b)

The hedge portfolio consists of long underlying bonds and short put option.

$$\Pi(r,t) = -V(r,t) + \Delta \times 100 \times Z(0,3),$$

Choose Δ making $\frac{\partial \Pi}{\partial r} = 0$. By differentiating w.r.t r,

$$\Delta = \frac{\partial V/\partial r}{100\ \partial Z(0,3)/\partial r} = \frac{66.3}{100 * B(0,3)Z(0,3)} = \frac{66.3}{100 * 1.62 * 0.8848} = 0.462$$

So it requires 0.462 of $Z(0,3)$. When $Z(0,3)$ is replicated by $Z(0,5)$ as in (b), it is

$$0.462 * (0.9288 * 79.55 + 14.60) = 0.43 * 79.55 + 6.75$$

Hence the portfolio of $Z(0,5)$ and cash is

$$\Pi(r,t) = -V(r,t) + 0.462 \times 100 \times 0.8848 = -4.1 + 0.43 * 79.55 + 6.75$$
$$= 36.85$$

Where:
The number of $Z(0,5) = 0.43$
Cash = 6.75
Option = -4.10 from part (a)

11. Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4e) Analyze the Greeks of common option strategies.

(4j) Compare and contrast "floating" and "sticky" smiles.

(4k) Describe and contrast several approaches for modeling smiles, including: stochastic volatility, local-volatility, jump-diffusions, variance-gamma, and mixture models.

Sources:

QFIQ-120-19 Pricing and Hedging of Financial Derivatives, Chapter 6, by Marroni and Perdomo

The Volatility Smile, Chapter 7, by Derman, Miller and Park, 2016

Commentary on Question:

This question tests candidates' understanding of delta hedging and volatility smile.

Solution:

(a) Calculate your cumulative total profit or loss on Day 4 under the following circumstances, respectively:

(i) You rebalanced your hedge position daily.

(ii) You never rebalanced your hedge position.

Commentary on Question:

Candidates performed below expectations in this part. Some were able to achieve full credit when they set up the Excel sheet correctly. Partial credit was awarded if the candidate's answer was correct for hedging of 1 option (rather than 1,000 options as asked by the question).

11. Continued

The hedge is to buy stocks when the call option is sold. Total hedge position consists of short calls and long stocks

I. With daily rebalancing: total gain = 459, as shown below:

Day = t	1	2	3	4	Total
Stock price $= S_t$	80	70	75	82	
Option price $= O_t$	12.25	12.25	12.22	12.30	
Option delta $= D_t$	0.610	0.535	0.562	0.638	
# of short options $= NO_t$	-1000	-1000	-1000	0	
# of long stocks $= NS_t = -NO_t * D_t$	610	535	562	0	
Gain from stocks $= GS_t = NS_{t-1} * (S_t - S_{t-1})$		-6100	2675	3934	509
Gain from options $= GO_t = NO_{t-1} * (O_t - O_{t-1})$		0	30	-80	-50
Total gain $= GS_t + GO_t$					**459**

II. Without daily rebalancing:
Total gain = 600 * (82 – 80) – 1000 * (12.30 – 12.25) = 1170

(b) Determine whether each of the three explanations provided is valid or not. Explain why.

Commentary on Question:
Candidates performed below expectations on this part. Only a small portion of candidates were able to justify why each of the analyst's three explanations is valid or not.

Overall:
Because there is no change of interest rate and the effect of the time decay over 1 day is small (in light of the option maturity of 3 years), the answer below ignores the effect of interest rate and time decay.

Observation 1
For a given strike, the "sticky strike rule" says that the implied vol does not change with the stock price. If this were true, the option price on Day 2 should have decreased due to decrease of the stock price. Since this is not the case, the sticky strike rule cannot explain Observation 1.

Observation 2
For a given strike, the "sticky delta rule" says that the implied vol increases when the stock price rises. If this were true, the option price on Day 3 should have increased due to increase of the stock price and the implied vol from Day 2 to Day 3. Since this is not the case, the sticky delta rule cannot explain Observation 2.

11. **Continued**

<u>Observation 3</u>
For a given strike, the local volatility model says that the implied vol falls when the stock price rises. This has two effects on the call option price: (i) when stock price rises, it increases the call option value; (ii) when the implied vol falls, it decreases the call option value. When these two effects happen simultaneously as in the local volatility model, it's possible for the option price to increase due to (i) outweighing (ii). So "local volatility model" can explain Observation 3.

(c) Provide your explanation for observation 4.

Commentary on Question:
Candidates performed as expected on this part.

Because there is no change in the stock price, interest rate and delta between Day 1 and Day 30, the decrease of the option price can be explained by option theta, or time decay.

12. Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4b) Identify limitations of the Black-Scholes-Merton pricing formula

(4c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(4e) Analyze the Greeks of common option strategies.

(4i) Define and explain the concept of volatility smile and some arguments for its existence.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a)

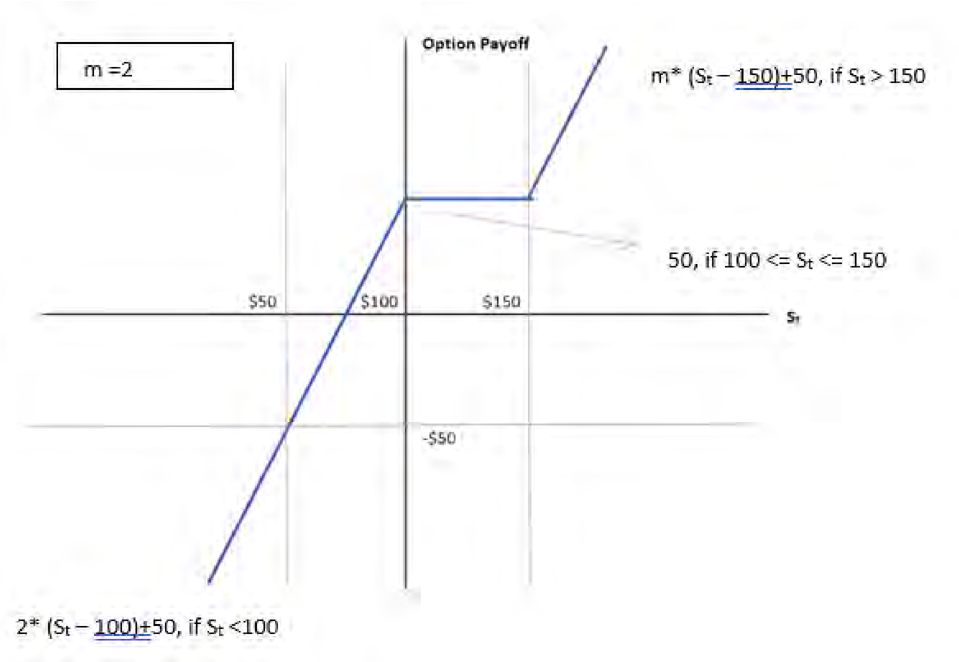
(i) Sketch the payoff graph for option E using $m = 2$.

(ii) Build a static hedging strategy with vanilla options to hedge the equity risk.

Commentary on Question:

Candidates did okay for this question. Most candidates were able to sketch the payoff and identify the embedded vanilla options. However, some candidates failed to recognize that the long position in call was subjected to equity risk. Some candidates were also confused about static hedging strategy with option vs. dynamic hedging with underlying assets, or failed to use the opposite position to hedge the portfolio.

12. Continued



ii)
m *(S – 150)+50, if 150 < S ;
0 , otherwise
= m* Max [S – 150, 0] + [50 | 150 < S]
→ Long m unit of call option with strike at $150 + [50 | 150 < S] - (1)

2 *(S – 100)+ 50, if S < 100;
0 , otherwise
= - 2* Max [100 - S, 0] + [50 | S < 100]
→ Short 2 unit of put option with strike at $100 + [50 | S < 100] - (2)

(1) + (2) + [50 | otherwise]
→ m * C(150) – 2 * P(100) + 50

Static Hedging Strategy: hold opposite postion to offset the equity exposure.

m=2
Short 2 unit of C(150) and long 2 unit of P(100)
Cash has no equity exposure

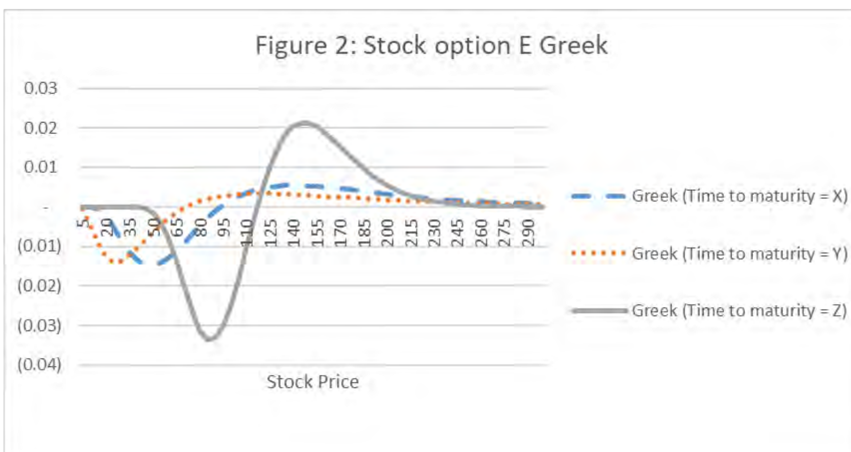
12. Continued

(b)

(i) Define the following Greeks: Delta, Gamma, Vega, and Theta.

(ii) Sketch Delta graph for option E using $m = 2$ and justify your answers. (Hint: Build from vanilla options.)

(iii) Determine which figure corresponds to Gamma, Vega, and Theta, respectively. Justify your answers.



12. Continued

Figure 3: Stock option E Greek

Commentary on Question:
Candidates did well for the part b(i).

For b(ii), most candidates knew that delta has to be positive (/ negative) for long position in call (/put). However, some didn't demonstrate their knowledge on how delta would behave when approaching the strike price, or got confused with the sign. In addition, some candidates didn't provide justification to support their answer, in that case, only partial credit is given.

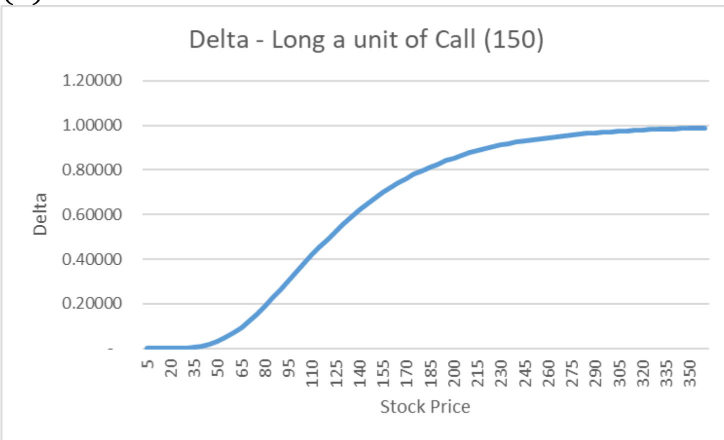
For b(iii), most candidates were able to identify the chart for theta and knew the sign for theta to be opposite from gamma and Vega. However, most candidates failed to recognize that gamma spikes up at ATM and Vega diminishes as time approaches the maturity date, and use it to differentiate the chart or use these as support for the graphs they identified as either gamma or Vega.

(i)

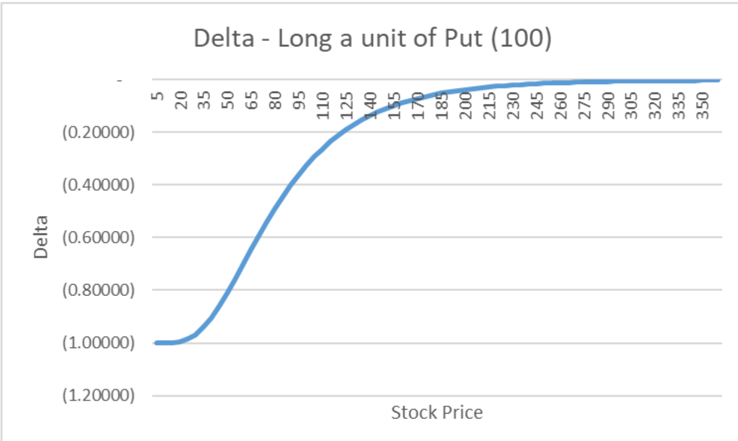
Delta	change in option price with relatively small changes in	price of underlying asset
Vega		volatility
Theta		passage of time.
Gamma	change in Delta with relatively small changes in	price of underlying asset

12. Continued

(ii)



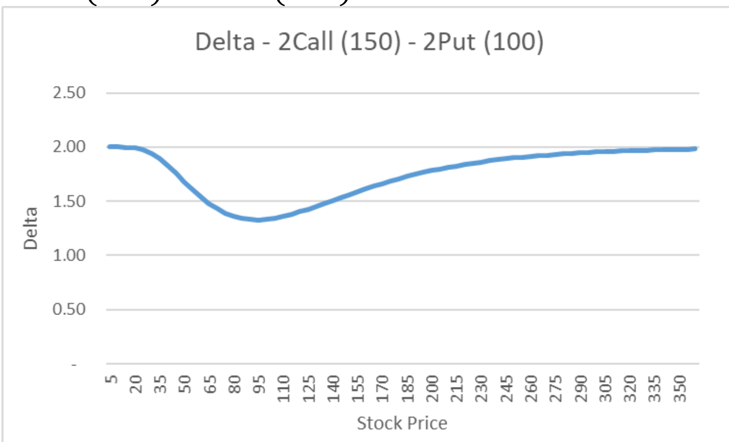
- Delta >0
- Delta grades from 0 (out of the money) to +1 (in the money)
- Delta is getting closer to 1 around $150.



- Delta <0
- Delta grades from 0 (out of the money) to -1 (in the money)
- Delta is getting closer to -1 around $100.

12. **Continued**

Cash has no delta, so the delta for option E is the delta for
2 Call (150) – 2 Put (100)



- Delta >0
- Delta <1, when stock price is around the range of (100, 150)
- Delta grades to 1 beyond 150, and beneth 100

(iii)
Figure 1 Vega
Figure 2 Gamma
Figure 3 Theta

Theta is negaive for a long positon of vallina option. Option E consists of a long call with strike price at 150 and a short put with strike price at 100.
→ negtive around $150, and positive around $100.
→ Only Figure 3 fits the profile.

Both Vega and Gamma are positive for a long position of vallina option.
→ positive around $150, and negative around $100.
→ Figure 1 and 2 fit the profile.

To differetiate Figure 1 vs. Figure 2:
Vega diminishes progressively with the reduction of time-to maturity regardless of the stock price level, which is not the case for Gamma.

- When the stock price is close to the strike price near the option expiry, a small change in stock price could quickly result in call option delta flipping between 0 and 1, that is, Gamma is highly unstable in this sitution.
- When the stock price is away from the strike price, Gamma dimishes with the reduction of the time-to maturity, similar to Vega.

→ Figure 1 is Vega, and Figure 2 is Gamma.

12. **Continued**

(c)

(i) Explain what the volatility skew is.

(ii) List three reasons why the volatility skew exists.

(iii) Explain why option *E* is not the suitable vehicle to trade on convexity of volatility skew.

Commentary on Question:
Most candidates could explain what volatility skew is and identify the high demand for OTM put against downside risks is one of the key drivers for volatility skew. However, most candidates failed to identify the other drivers.

For part ii), only some of candidates were able to identify that option E was equivalent to risk reversal with cash position, and was unsuitable to trade on convexity of volatility skew.

(i)
Volatility skew is a form of volatility smile, describing the relationship between implied volatilities (BSM) and strikes, where that downside strikes have greater implied volatility than upside strikes. The graph of implied volatility vs. strike price is thus showing a skew type of shape

Reason to have volatility skew:

- Demand component. Investors who own equities may want to hedge against large losses, and willing to pay extra (/risk premium) for the protections/ insurance against the risks of extrem events.
- Risk premium to comensate option sellers for the negative vega convexity that they take on from selling the out of the money options. ie. When the levels of volaitlity increases, the trader's negative vega position also increases. Seller than need to buy at higher volatility to rebalance, which cost more.
- Out of the money options are likely less liquid than at/in the money options, and thus harder to hedge.
- The risk of the price of the underlying asset having sudden/immediate jump is much larger for a deeply out of the money option than what a seller would take with an at the money options.

12. Continued

(ii)
Option E takes long vega position at strike price of $150, and short vega position at $100, which is equivalent to a risk reversal strategy. With all others being equal, traders can benefit from any increase in the implied volatility at the strike price of $150 relative to the strike price of $100.

To trade on volatiltiy convexity, traders need to long vega at OTM strikes on both sides, and short vega near the ATM strike (butterfly strategy), to benefit from the change in implied volatility becoming more (less) at OTM strike than ATM strike.

Option E is a vehicle to trade on the steepness/flateness of volatility skew, but not convexity.

(d)

(i) Determine K^* for option E^*.

(ii) Solve for m so that option E^* is Vega-neutral.

Commentary on Question:
Many candidates didn't attempt the question, but for those who attempted the question, most knew to set k = 50 to build a butterfly strategy, and tried to find m to achieve Vega-neutral.*

Most candidates assumed that m= 2 for part d), which is incorrect, but majority of the credit was still given if candidates performed the rest of the calculations correctly.

For candidates who didn't provide proper justification to support their answer, partial credit was given.

To trade on the convexity of volatility skew and benefit from increase in convexity, option E* needs to long Vega at the OTM strike and short/neutralize Vega near the ATM strike.

Option E* consists of

- 2 units of short call at strike = $100 (negative Vega)
- m units of short(/or long) put at strike = $k*
- Option E:

m units of long call at strike = $150 (positive Vega)
+ 2 units of short put at strike = $100 (negative Vega)

12. **Continued**

The strike price $k* needs to be OTM.

To have option E* with symmetric payoff centered at the current stock price, we need to have k* = $50 in long position, and have the same number of call at strike = $150 in long position.

- 2 unit of short call at \$100 + 2 unit of short put at \$100
  ➔ already have symmetric payoff centered at the current stock price
- m units of long call at \$150 +m units of short(/or long) put at \$50
  ➔ to be symmetric

To achieve Vega neutral,

$$\frac{\{Vega;\ 2(Short\ Put\ (100) + Short\ Call\ (100))\}}{\{Vega;\ m(Long\ Call\ (150) + Long\ Put\ (50))\}} = -1$$

Calculate Vega $= SN'(d_1)\sqrt{T-t}$

Vega on Put (100)= Vega on Call (100)
= 100* 0.36014 * sqrt (5) = 80.53

Vega on Call (150)
= 100 * 0.36718 *sqrt(5) = 82.1

Vega on Put (50)
= 100 *0.11236 * sqrt(5) = 25.12

$$\frac{\{Vega;\ 2\big(Short\ Put\ (100) + Short\ Call\ (100)\big)\}}{\{Vega; N\big(Long\ Call\ (150) + Long\ Put\ (50)\big)\}}$$
$$= \frac{-2 * (80.53 + 80.53)}{m * (82.1 + 25.12)} = \frac{-322.12}{107.22\, m}$$

$$Set\ \frac{-322.12}{107.22\, m} = -1 \;=>\; m = 3$$

12. Continued

(e)

(i) Calculate the gain or loss of option E^*.

(ii) Demonstrate how option E^* is an effective vehicle to take position on volatility convexity, given the result in part (e)(i).

Commentary on Question:
Most candidates didn't attempt the question, but for those who did, most knew that the gain for option E should be calculated by reflecting the stock price change, and comment on the effectiveness of the option taking position on volatility convexity.*

(i)
Option E*= 3 * C(150) +3 * P (50) – 2 * P(100) – 2 * C(100)

The inital price of Option E*:

$C(150) = S * N(d_1)_{150} - 150 \exp(-2\% * 5) * N(d_2)_{150}$
$= 100 * 0.3419 - 150 \exp(-2\% * 5) * 0.1891$
$= 8.5242$

$C(100) = S * N(d_1)_{100} - 100 \exp(-2\% * 5) * N(d_2)_{100}$
$= 100 * 0.6745 - 100 \exp(-2\% * 5) * 0.4728$
$= 24.6693$

$P(100) = 100 \exp(-2\% * 5) * N(-d_2)_{100} - S * N(-d_1)_{100}$
$= 100 \exp(-2\% * 5) * 0.5272 - 100 * 0.3255$
$= 15.1530$

$P(50) = 50 \exp(-2\% * 5) * N(-d_2)_{100} - S * N(-d_1)_{50}$
$= 50 \exp(-2\% * 5) * N(-0.97) - 100 * N(-1.59)$
$= 50 \exp(-2\% * 5) * 0.1651 - 100 * 0.0557$
$= 1.8994$

→ 3 * 8.5242 + 3 * 1.8994 – 2 * 15.1530 – 2 * 24.6693 = – 48.3735

The price of Option E' after stock price decreases from 100 to 80:

$C(150)^* = S * N(d_1)^*_{150} - 150 \exp(-2\% * 5) * N(d_2)^*_{150}$
$= 80 * 0.4315 - 150 \exp(-2\% * 5) * 0.1486$
$= 14.3512$

12. Continued

$C(100)^* = S * N(d_1)^*_{100} - 100 \exp(-2\% * 5) * N(d_2)^*_{100}$
$= 80 * 0.5603 - 100 \exp(-2\% * 5) * 0.3019$
$= 17.5070$

$P(100)^* = 100 \exp(-2\% * 5) * N(-d_2)^*_{100} - S * N(-d_1)^*_{100}$
$= 100 \exp(-2\% * 5) * 0.6981 - 80 * 0.4397$
$= 27.9907$

$P(50)^* = 50 \exp(-2\% * 5) * N(-d_2)^*_{100} - S * N(-d_1)^*_{50}$
$= 50 \exp(-2\% * 5) * 0.3850 - 100 * 0.1341$
$= 6.6901$

$\rightarrow 3 * 14.3512 + 3 * 6.6901 - 2 * 17.5070 - 2 * 27.9907 = -27.8715$

Gain on Option E*
= The price of option E* $_{\text{after S: 100 ->80}}$ – The price of option E* $_{\text{Initial}}$
= 20.502

(ii)
Following the decrease in stock price from 100 to 80, we have increase in implied volatility where the increase is more significant at out of the money strike than at the money strike.

Option E* long Vega at out of the money strike and short at at the money strike, and thus generates gains benefiting from the increase in volatility convexity

13. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

Commentary on Question:

This question tests candidates' understanding of risk mitigation techniques and how to apply these techniques in a hypothetical situation. The question tests candidates' knowledge of the operational definition of metrics associated with risk and how they change with changes to the risk profile of a portfolio.

Solution:

(a)

(i) Describe the following risk mitigating methods: dilution, diversification, and hedging.

(ii) Identify the situations where each method would be efficient.

Commentary on Question:

Candidates performed well in reciting the operational definitions of the three methods, with extremely strong performance for diversification and hedging. Identification of the situations where each method would be efficient had relatively strong performance for diversification and hedging and relatively poor performance for dilution.

(i) Dilution – Increasing the investment in the riskless bond so that the proportion of the portfolio that is comprised of the riskless bond is much higher in relation to the proportion of the risky assets.

Diversification - Investing in a large number of stocks.

Hedging – Investing in securities that are correlated to a risk.

13. Continued

(ii) Dilution – efficient when many riskless assets are available.

Diversification – more efficient when there are many securities available for investment that have low correlations with one another.

Hedging – more efficient when there are securities available that are correlated to the risk that is to be hedged.

(b) Explain the effect of diluting risks on the Sharpe ratio.

Commentary on Question:
This question tested the application of the Sharpe ratio to a situation where there is a change in the portfolio composition. Most candidates recited the formula for the Sharpe ratio and stated that there would be no change. Few candidates justified the conclusion with the derivation of the Sharpe ratio for the revised portfolio.

Expected return of the diluted portfolio is

$$\mu_P = w\mu + (1-w)r = r + w(\mu - r),$$

where μ is the return rate of stock S , r is the riskless rate, w is the weight of the risky asset after the dilution through the investment in the riskless asset.

The volatility of the portfolio is $\sigma_P = w\sigma + 0 = w\sigma$, where σ is the volatility of stock S.

The Sharpe ratio of the portfolio is $\lambda_P \equiv \frac{\mu_P - r}{\sigma_P} = \frac{r + w(\mu-r) - r}{w\sigma} = \frac{w(\mu-r)}{w\sigma} = \frac{\mu - r}{\sigma}$
This is the same as the Sharpe ratio of the stock, which indicates that there will be no effect on the Sharpe ratio.

Commentary on Question:
Few candidates demonstrated an understanding of the structure that replicates the market risk neutral stock under the premise that all stocks in the given investment universe are correlated with the market.

As the stocks are all correlated with the market, then a market risk neutral stock S'_i can be created by shorting exactly Δ_i shares of M against S_i, where:

$$\Delta_i = \rho_i \frac{\sigma_i}{\sigma_M} \frac{S_i}{M} = \beta_i \frac{S_i}{M},\ \beta_i = \rho_i \frac{\sigma_i}{\sigma_M}.$$

Therefore, $S'_i = S_i - \Delta_i M = S_i - \beta_i \frac{S_i}{M} M = (1 - \beta_i) S_i$.

13. Continued

(d) Calculate the expected market return μ_M, and the Sharpe ratio of any market-neutral portfolio S_i'.

Commentary on Question:
This question tests the candidates' understanding of the relationship between the expected return of the market risk neutral portfolio and the riskless bond. Many candidates demonstrated their knowledge of the Sharpe ratio and few candidates applied the values properly when calculating the expected market return.

The portfolio will replicate a riskless bond with number of market risk neutral stocks S'_i goes to infinity. This indicates that the expected return of each market risk neutral stock S'_i must have the same return rate as the riskless rate r.

Therefore, the Sharpe ratio of $S'_i = \frac{\mu'_i - r}{\sigma'_i} = \frac{r-r}{\sigma'_i} = 0$.

and,

$$\mu_i - r = \beta_i(\mu_M - r).$$

The expected market return is therefore

$$\mu_M = \frac{\mu_i - r}{\beta_i} + r = \frac{8\% - 2\%}{3} + 2\% = 4\%.$$

14. Learning Objectives:

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(5b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

Sources:

QFIQ-133-21: IAA, Stochastic Modeling, Theory and Reality from and Actuarial Perspective., Section IV.A.1-8

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a)

(i) Identify the option embedded in the VA with GLWB rider.

(ii) Explain when the option in part (a)(i) will be in-the-money (ITM).

Commentary on Question:

Most candidates could identify the embedded option; however, majority could not explain under what situation the option is in-the-money.

(i) The GLWB effectively represents an embedded put option to the policyholders, because payments are made to them even if the account value is depleted.

(ii) The option is ITM if PV of GLWB is greater than AV. The option is in the money as the account value diminishes or is depleted, the insurance company is liable for any future claims (withdrawals) that must be paid until the policyholder dies.

(b) Critique the above approach with respect to the following aspects:

(i) The use of risk-neutral scenarios

(ii) The number of scenarios

(iii) The methodology of calculating the cost of GLWB

14. Continued

Commentary on Question:
In general, candidates did well on (i) and (ii), but could not articulate why the GLWB claim may not be zero at the end of the 30-year period.

(i) Use of risk-neutral scenario:
- For the pricing of the cost of the embedded option, risk-neutral scenarios are typically appropriate.
- The real-world scenarios are more likely to provide information for understanding the risk of the entire VA contract than risk-neutral scenarios.
- Since the GLWB option by nature is an equity derivative, a claim is triggered when account value drops to zero. If the guarantee involves a minimum guaranteed interest rate, stochastic interest rate model would be required.

(ii) Number of Scenarios
- 1000 scenarios seem appropriate
- Common practice is to use 1000 in US
- Study shows a large deviation occurs when scenarios number is less than or equal to 200

(iii) Cost GLWB
- If the account value stays positive at the end of year 30, the present value of annual guaranteed benefit over the policyholder's remaining life should be compared with the account value. If the present value is higher, the difference is an approximate of the GLWB claims. Otherwise, the GLWB is zero.

(c) Assess and justify the reasonableness of the dynamic lapse factors of the base and sensitivity assumptions.

Commentary on Question:
Candidates did well on the first part, i.e. when the option is in-the-money, policyholders are more likely to retain the coverage.

However, most candidates did not recognize that the lapse assumptions for the sensitivity test are higher than the base when the option is in-the-money, this does not reflect an adverse scenario.

Under both the assumed/base and sensitivity assumptions, policyholders are more likely to retain the coverage with the guarantee is valuable/ITM. This is reasonable.

14. **Continued**

The sensitivity assumption's lapse factor is relatively higher compared to the base assumption. This is not reasonable for sensitivity testing, as the sensitivity assumption is less adverse than the base assumption.

(d) Explain and justify whether you agree with the sensitivity test results.

Commentary on Question:
Majority of the candidates did not demonstrate the comprehension of the fact that, with the higher lapse assumptions of the sensitivity test, the sensitivity test should produce lower, not higher, cost for GLWB.

The results are flipped. Under the sensitivity assumption, the lapse is higher which implies that the policyholders are more likely to terminate the policy even when it is in-the-money (ITM). When this happens, the insurance company's liability is reduced, and thus the cost of the GLWB (under sensitivity assumption) should be lower.

(e) Your company is considering the addition of a new equity fund (albeit aggressive growth). Existing funds are one conservative equity fund, one moderate growth equity fund and a bond fund. The volatility of the bond fund is lower than the conservative equity fund. The table below summarizes the sensitivity test results of replacing the moderate growth equity fund with the new equity fund assuming other fund allocation assumptions remain the same:

Sensitivity test results of fund replacement
(Average cost of GLWB % of Benefit Base)

Age	Base	Sensitivity
Under 65	78.7 bps	92.7 bps
65 and above	48.7 bps	58.6 bps

Draw a conclusion from the above results.

Commentary on Question:
Most candidates did not draw the accurate conclusion.

While the fund allocation is assumed to be same, the cost of GLWB increases for both age groups, we can conclude that the new fund return is more volatile than the old fund.

15. **Learning Objectives:**

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(5b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

Sources:

QFIQ-128-20: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Commentary on Question:

A significant number of candidates did not attempt this question. The commentary below only considers the candidates who attempted to answer the question

Solution:

(a) Show that the value of the liability guarantee is
$\Omega_t = \Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)[0.99 * (0.9635 + 0.015 * ITM)]^{T-t}, \quad if\ 1.1 < ITM < 2.3$

Commentary on Question:

No credit was given to candidates who simply wrote the equation they were trying to show. Among candidates who attempted the question, roughly 25% either skipped part (a) or got no credit. Better answers showed clearly the substitution process below with a clear explanation of the pieces.

$$\Omega_t = \Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)\,{}_{T-t}p_{x+t}$$

${}_{T-t}p_{x+t} = {}_{T-t}p_{x+t}^{(d)}\,{}_{T-t}p_{x+t}^{(w)}$, where

${}_{T-t}p_{x+t}^{(d)} = [1 - 0.01]^{T-t}$, and

${}_{T-t}p_{x+t}^{(w)} = \left[\left(1 - .02 * \left(1 - 0.75 * (ITM - 1.1)\right)\right)\right]^{T-t}$

$=\left[(1 - 0.01)\left(1 - .02 * \left(1 - 0.75 * (ITM - 1.1)\right)\right)\right]^{T-t}$

$= \left[(0.99)\left(1 - .02 * (1.825 - 0.75 * ITM)\right)\right]^{T-t}$

$= [(0.99)(0.9635 + .015 * ITM)]^{T-t}$

15. Continued

Thus,

$\Omega_t = \Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)\,{}_{T-t}p_{x+t}$

$= \Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)[(0.99)(0.9635 + .015 * ITM)]^{T-t}$

(b) Show that, for $1.1 < ITM < 2.3$, the Delta of the value of the liability guarantee is:

$$\frac{\partial \Omega_t}{\partial A_t} = -e^{-a(T-t)}\Phi(-d_1)\frac{\Omega_t}{\psi(t, T, A_t, G_T)} - (T-t)\left(\frac{\Omega_t(0.015 \cdot ITM)}{A_t\left[(0.9635 + 0.015 \cdot ITM)\right]}\right)$$

Commentary on Question:

Generally candidates who attempted this question did well on the application of the product rule. The second term was more difficult and that was reflected in the responses.

Apply product Rule:

$$\frac{\partial \Omega_t}{\partial A_t} = \frac{\partial \Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)}{\partial A_t}\,{}_{T-t}p_{x+t} + \Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)\frac{\partial\,{}_{T-t}p_{x+t}}{\partial A_t}$$

Now for the first term:

From part a: $\Omega_t = \Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)\,{}_{T-t}p_{x+t}$ or ${}_{T-t}p_{x+t} = \frac{\Omega_t}{\Psi(\mathrm{t},\mathrm{T},\mathrm{A}_t,\mathrm{G}_T)}$

Also given is that $\frac{\partial \Psi(\mathrm{t},\mathrm{T},\mathrm{A}_t,\mathrm{G}_T)}{\partial A_t} = -\mathrm{e}^{-a(\mathrm{T}-\mathrm{t})}\Phi(-d_1)$

For the second term:

From part a: ${}_{T-t}p_{x+t} = [0.99 * (0.9635 + 0.015 * \frac{\mathrm{G}_t}{\mathrm{A}_t})]^{T-t}$ since $\frac{\mathrm{G}_t}{\mathrm{A}_t} = ITM$

$$\frac{\partial\,{}_{T-t}p_{x+t}}{\partial A_t} = (T-t)\left[0.99 * \left(0.9635 + 0.015 * \frac{\mathrm{G}_t}{\mathrm{A}_t}\right)\right]^{T-t-1}\left(-0.99 * 0.015\frac{\mathrm{G}_t}{\mathrm{A}_t^2}\right)$$

$$= \frac{(T-t)\left[0.99 * \left(0.9635 + 0.015 * \frac{\mathrm{G}_t}{\mathrm{A}_t}\right)\right]^{T-\mathrm{t}}\left(-0.99 * 0.015\frac{\mathrm{G}_t}{\mathrm{A}_t^2}\right)}{[0.99 * \left(0.9635 + 0.015 * \frac{\mathrm{G}_t}{\mathrm{A}_t}\right)]}$$

Substituting back $ITM = \frac{\mathrm{G}_t}{\mathrm{A}_t}$ and : ${}_{T-t}p_{x+t} = [0.99 * (0.9635 + 0.015 * \frac{\mathrm{G}_t}{\mathrm{A}_t})]^{T-t}$

$$= -(T-t)\,\frac{{}_{T-t}p_{x+t}ITM(0.015)}{\mathrm{A}_t[(0.9635 + .015 * ITM)]}$$

Thus $\Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)\frac{\partial\,{}_{T-t}p_{x+t}}{\partial A_t} = -(T-t)\,\Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)\frac{{}_{T-t}p_{x+t}ITM(0.015)}{\mathrm{A}_t[(0.9635 + .015 * ITM)]}$

15. Continued

And finally substituting $\Omega_t = \Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)_{T-t}p_{x+t}$ has the second term as:

$$-(T-t)\left(\frac{\Omega_t(0.015 * ITM)}{A_t[(0.9635 + 0.015 * ITM)]}\right)$$

Combining them gives the final equation:

$$= -\mathrm{e}^{-a(\mathrm{T-t})}\Phi(-d_1)\frac{\Omega_t}{\Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)} - (T-t)\left(\frac{\Omega_t(0.015 * ITM)}{A_t[(0.9635 + 0.015 * ITM)]}\right)$$

as desired.

(c) Determine, for $1.1 < ITM < 2.3$ using results from part (b), whether the absolute magnitude of the Delta of the value of the liability guarantee is larger with or without dynamic lapse multiple.

Commentary on Question:
Almost no candidates made any attempt at this problem. Partial credit was awarded to candidates who attempted to reason logically which way the sensitivity should have gone.

Suppose that we denote $\frac{\partial \Omega_t^*}{\partial A_t}$ as the delta of the value of the liability guarantee if the lapse assumption does not have dynamic lapse multiple (i.e., the lapse rate is the base lapse of 2%/year).

Without dynamic lapse multiple, the delta is

$$\frac{\partial \Omega_t^*}{\partial A_t} = -\mathrm{e}^{-a(\mathrm{T-t})}\Phi(-d_1)(0.99)^{T-t}(0.98)^{T-t}$$

With dynamic lapse multiple, from part b), the delta is $(for\ 1.1 < ITM < 2.3)$

$$\begin{aligned}\frac{\partial \Omega_t}{\partial A_t} &= -\mathrm{e}^{-a(\mathrm{T-t})}\Phi(-d_1)\frac{\Omega_t}{\Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)} - (T-t)\left(\frac{\Omega_t(0.015 * ITM)}{A_t[(0.9635 + 0.015 * ITM)]}\right) \\ &= -\mathrm{e}^{-a(\mathrm{T-t})}\Phi(-d_1)[0.99 * (0.9635 + 0.015 * ITM)]^{T-t} \\ &\quad - (T \\ &\quad - t)\left(0.99\frac{0.015 * ITM}{A_t}\right)\Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)[0.99 \\ &\quad * (0.9635 + 0.015 * ITM)]^{T-t-1}\end{aligned}$$

Suppose we denote

$$X = \mathrm{e}^{-a(\mathrm{T-t})}\Phi(-d_1)[0.99 * (0.9635 + 0.015 * ITM)]^{T-t}$$

$$Y = \left(0.99\frac{0.015 * ITM}{A_t}\right)\Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)[0.99 * (0.9635 + 0.015 * ITM)]^{T-t-1}$$

Then, $\frac{\partial \Omega_t}{\partial A_t} = -X - (T-t)Y$

15. **Continued**

So, we will show that

1) X is greater than the absolute magnitude of $\frac{\partial\Omega_t^*}{\partial A_t}$
2) $Y \geq 0$

Note that

$$X = \mathrm{e}^{-a(\mathrm{T-t})}\Phi(-d_1)[0.99 * (0.9635 + 0.015 * ITM\%)]^{T-t} \geq \left|\frac{\partial\Omega_t^*}{\partial A_t}\right|$$
$$= \mathrm{e}^{-a(\mathrm{T-t})}\Phi(-d_1)(0.99)^{T-t}(0.98)^{T-t}$$

(Note : $|x|$ = absolute value of x)

because

$(0.9635 + 0.015 * ITM)^{T-t} \geq (0.98)^{T-t}\ for\ 1.1 < ITM < 2.3$

Moreover, the following expression is non-negative

$$Y = 0.99\left(\frac{0.015 * ITM}{A_t}\right)\Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T)[0.99 * (0.9635 + 0.015 * ITM)]^{T-t-1}$$

Because ITM > 0 and $\mathrm{A}_t \geq 0$ and $\Psi(\mathrm{t}, \mathrm{T}, \mathrm{A}_t, \mathrm{G}_T) \geq 0$

Therefore, the conclusion is that the absolute magnitude/value of $\frac{\partial\Omega_t}{\partial A_t}$ is <u>greater with the dynamic lapse multiple.</u>