An Empirical-based Approach for Optimal Reinsurance

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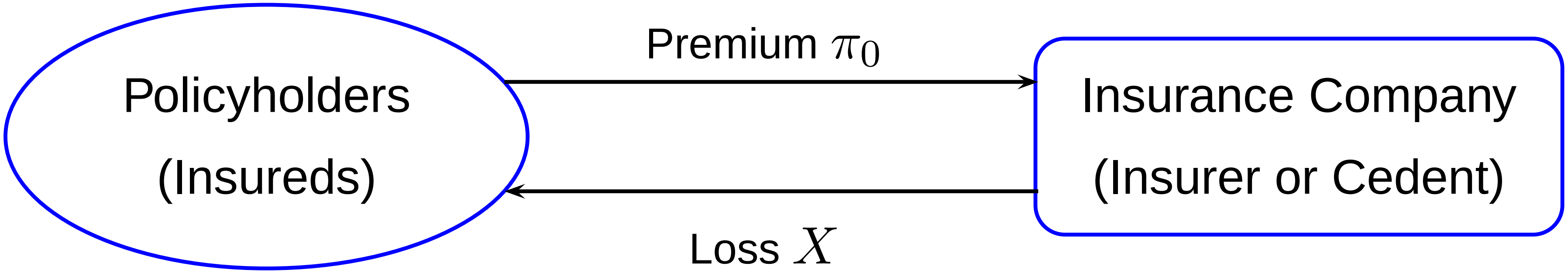
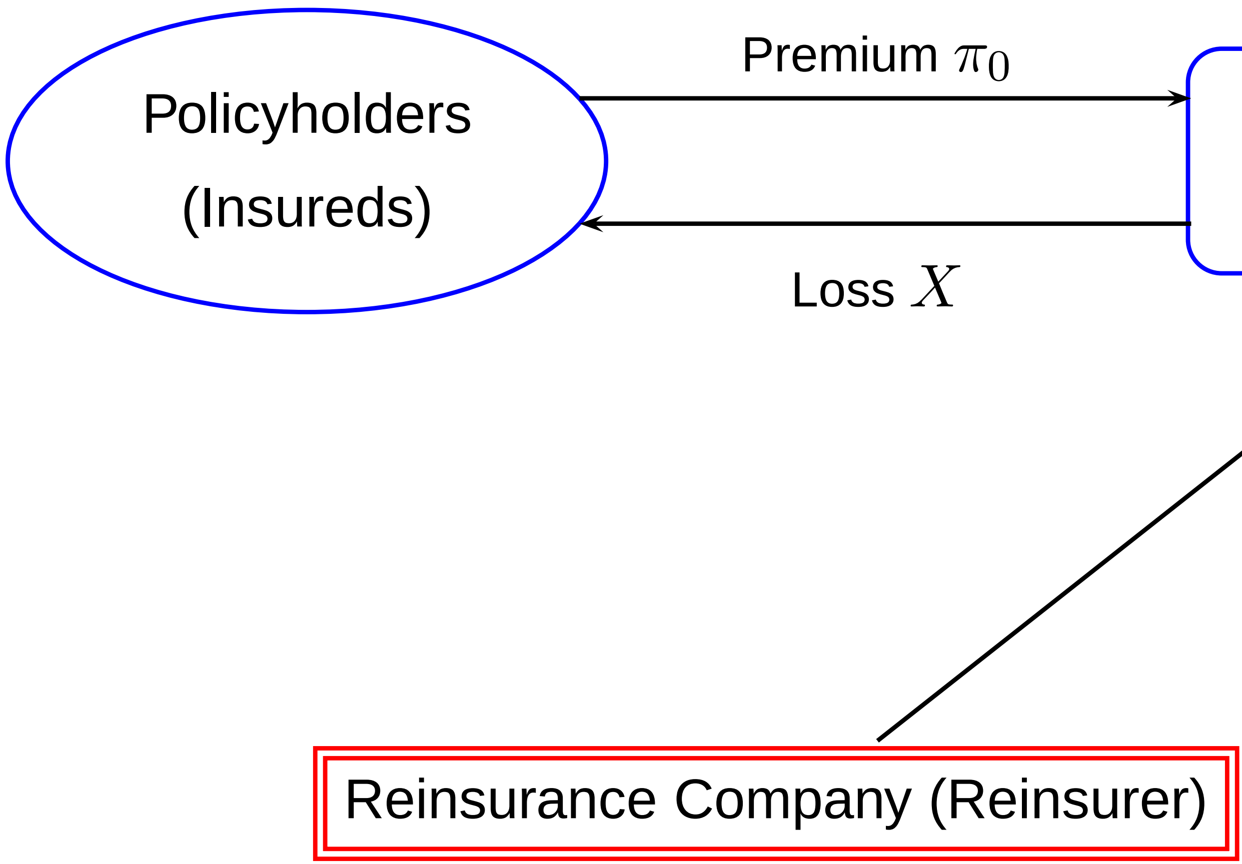
Outline of Today's Presentation

- Background
- Motivation
- Empirical-based Approach
- Conclusion

Effect of Reinsurance



Effect of Reinsurance



$f(X)$: Ceded Claims

e.g. stop loss:

$$f(X) = (X - d)_+, d \geq 0$$

e.g. quota-share:

$$f(X) = cX, 0 \leq c \leq 1$$

Effect of Reinsurance

Policyholders (Insureds) — Premium π_0 → Insurance Company (Insurer or Cedent)

Insurance Company (Insurer or Cedent) — Loss X → Policyholders (Insureds)

$\Pi(f)$: Reinsurance Premium

e.g. Expectation premium principle:

$$\Pi(f) = (1+\theta)\mathrm{E}[f(X)]$$

$\Pi(f)$

$f(X)$: Ceded Claims

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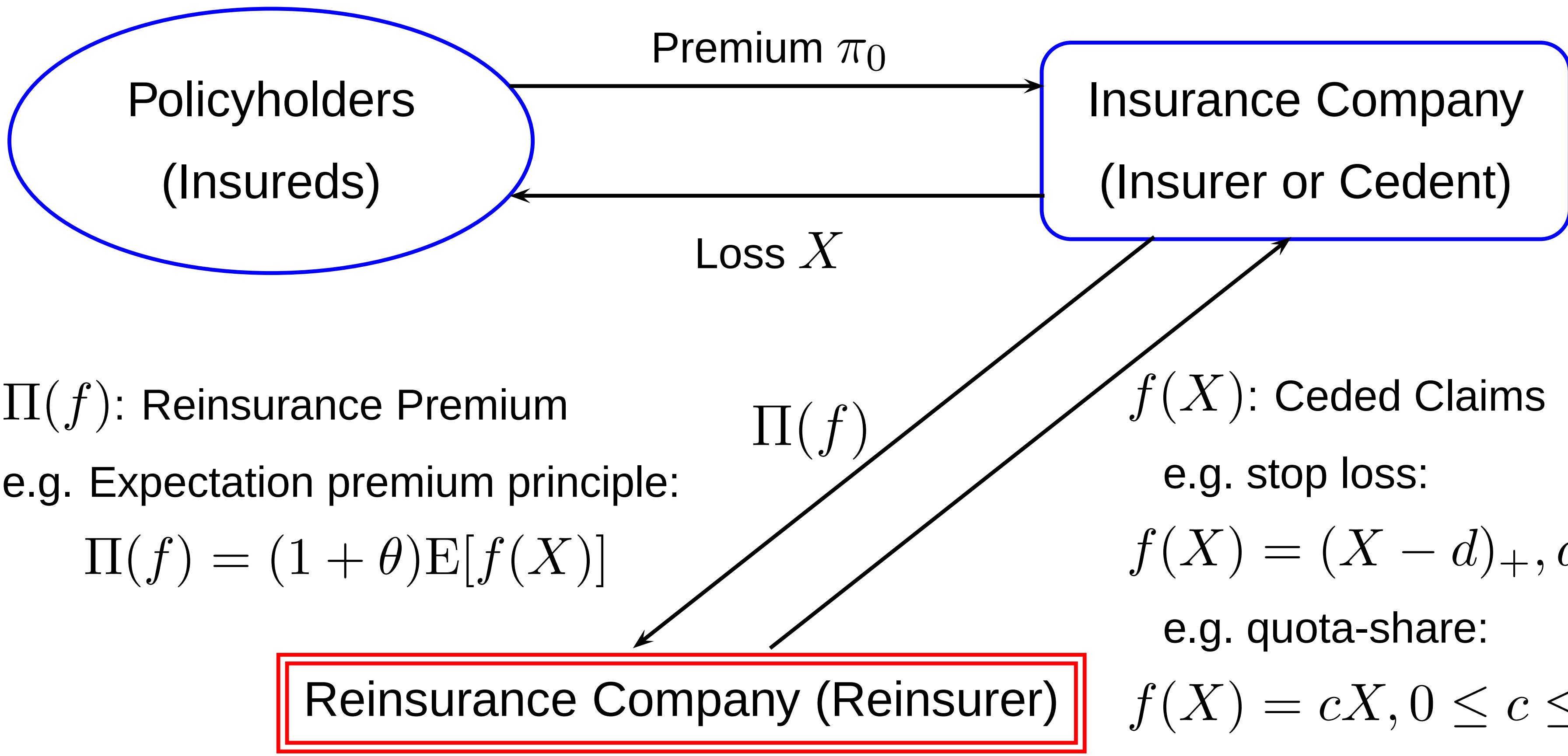
Reinsurance Company (Reinsurer)

Effect of Reinsurance



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- Insurer's retained risk: $R_f(X) = X - f(X)$
- Insurer's total risk: $T_f(X) = R_f(X) + \Pi(f) = X - f(X) + \Pi(f)$
- **tradeoff** between the amount of loss retained and the reinsurance premium payable to a reinsurer

Risk Measure Minimization Reinsurance Models

- A plausible optimal reinsurance model:

$$\begin{aligned} &\min_f \quad \rho(T_f(X)) = \rho\big(X - f(X) + \Pi(f(X))\big) \\ &\text{s.t.} \quad \Pi(f(X)) \le \pi \\ &\qquad 0 \le f(x) \le x \text{ for all } x \ge 0 \end{aligned}$$

  - where ρ is a chosen risk measure, such as variance, VaR and CTE.

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  - where ρ is a chosen risk measure, such as variance, VaR and CTE.
- Complexity of solving the above risk measure minimization models:
  - if the form of f is specified: technically tractable.
    - stop-loss $f(x) = (x - d)_+$
    - quota-share $f(x) = cx$
  - if a general f is considered: infinite dimensional problem, very challenging to obtain the explicit solutions.
  - relies on the premium principle and the risk measure ρ.

Empirical Approach: Motivation

- The general reinsurance model can be formulated as:

$$\begin{cases} \min_f & \rho(X, f) \\ \text{s.t.} & 0 \le f(x) \le x \text{ for all } x \ge 0, \\ & \Pi(f(X)) \le \pi. \end{cases}$$

  - Often difficult to solve (due to infinite dimension and nonlinearity)

- In practice, the distribution of the underlying risk X is estimated from the observed data $\{x_1, \cdots, x_N\}$.

- **Empirical-based reinsurance model**:
  - exploits the observed data directly

Empirical-based Model: Formulation

- Collect samples $\mathbf{x} := (x_1, x_2, \cdots, x_N)$ corresponding to the underlying risk X

- Introduce decision vector $\mathbf{f} := (f_1, f_2, \cdots, f_N)$ where each f_i represents the reinsurance indemnification for the loss amount x_i, $i = 1, 2, \cdots, N$.

- Define empirical-based estimates as:

$$\begin{array}{lcl} \rho(X, f) & \longrightarrow & \widehat{\rho(\mathbf{x}, \mathbf{f})} \\ 0 \leq f(x) \leq x, \ \text{for all } x \geq 0 & \longrightarrow & 0 \leq f_i \leq x_i, \ \ i = 1, 2, \cdots, N, \\ \Pi(f) \leq \pi & \longrightarrow & \widehat{\Pi(\mathbf{f})} \leq \pi \end{array}$$

Empirical-based Model: Formulation (Cont'd)

- General reinsurance model

$$\begin{cases} \min_{\mathbf{f}} & \rho(\mathbf{x}, \mathbf{f}) \\ \text{s.t.} & 0 \le f(x) \le x, \ \text{ for all } x \ge 0, \\ & \Pi(\mathbf{f}) \le \pi. \end{cases}$$

- Empirical reinsurance model:

$$\begin{cases} \min_{\mathbf{f}} & \widehat{\rho(\mathbf{x}, \mathbf{f})} \\ \text{s.t.} & 0 \le f_i \le x_i, \ \ i = 1, 2, \cdots, N, \\ & \widehat{\Pi(\mathbf{f})} \le \pi. \end{cases}$$

- Many empirical reinsurance model can be cast as **Second-Order Conic (SOC)** programming:
  - *A wide class of optimization problems*
  - *Efficient softwares are available for solving SOC programming: e.g., CVX (Grant and Boyd, 2008)*

Empirical Model: Variance Minimization

- Consider the following variance minimization model:

$$\begin{cases} \min_f & \mathsf{Var}(T_f) = \mathsf{Var}(X - f(X) + \Pi(f)) \\ \text{s.t.} & 0 \le f(x) \le x, \;\; \Pi(f) \le \pi. \end{cases}$$

- Empirical version of the goal function:

$$\widehat{\mathsf{Var}(T_f)} = \frac{1}{N-1} \sum_{i=1}^{N} \left[\left(x_i - f_i \right) - (\bar{x} - \bar{f}) \right]^2,$$

where $\bar{x}$ denotes the mean of $\mathbf{x}$, and $\bar{f}$ denotes the mean of $\mathbf{f}$.

- Empirical version of the constraints:

$$0 \le f_i \le x_i, \;\; i = 1, 2, \cdots, N, \;\; \text{and} \;\; \widehat{\Pi(\mathbf{f})} \le \pi.$$

- Empirical variance minimization model:

$$\begin{cases} \min_{\mathbf{f} \in \mathbb{R}^N} & \sum_{i=1}^{N} \left[\left(x_i - f_i \right) - (\bar{x} - \bar{f}) \right]^2 \\ \text{s.t.} & 0 \le f_i \le x_i, \;\; i = 1, 2, \cdots, N. \\ & \widehat{\Pi(\mathbf{f})} \le \pi. \end{cases}$$

Reinsurance Premium Budget Constraint

- P1. Expectation principle: $\Pi(f) = (1+\theta)\mathsf{E}[f]$ with $\theta > 0$.

$$\widehat{\Pi(\mathbf{f})} \le \pi \iff (1+\theta)\bar{f} \le \pi,$$

- P2. Standard deviation principle: $\Pi(f) = \mathsf{E}[f] + \beta\sqrt{\mathsf{Var}[f]}$, where $\beta > 0$.

$$\widehat{\Pi(\mathbf{f})} \le \pi \iff \bar{f} + \frac{\beta}{\sqrt{N-1}}\left[\sum_{i=1}^{N}\left(f_i - \bar{f}\right)^2\right]^{1/2} \le \pi.$$

- ☐ *The empirical variance minimization model can be cast as a SOC programming problem for as many as ten principles.*
- ☐ *We analyzed the empirical solutions by some numerical examples (under expectation and std premium principles).*

Empirical Model: CTE minimization

- Theoretical CTE minimization model:

$$\begin{cases} \min_f & \text{CTE}_\alpha(T_f) \equiv \text{CTE}_\alpha\Big(X - f(X) + \Pi[f(X)]\Big) \\ \text{s.t.} & 0 \le f(x) \le x, \ \ \Pi[f(X)] \le \pi, \end{cases}$$

- Technical model:

$$\begin{cases} \min_{(\xi,f)} & G_\alpha(\xi, f) \equiv \xi + \dfrac{1}{\alpha}\mathsf{E}\left[\Big(X - f(X) + \Pi(f(X)) - \xi\Big)^+\right]. \\ \text{s.t.} & 0 \le f(x) \le x, \ \ \Pi(f(X)) \le \pi. \end{cases}$$

- We developed the following fact:
  - $(\xi^*, f^*) \in$ arg $\min_{(\xi,f)} G_\alpha(\xi, f)$ if and only if

$$f^* \in \text{arg}\ \min_f \text{CTE}_\alpha(T_f), \quad \xi^* \in \text{arg}\ \min_\xi G_\alpha(\xi, f^*).$$

- We proved that stop-loss is the optimal solution given that Π is the expectation principle.

Empirical Model: CTE minimization (Cont'd)

- Empirical model:

$$\begin{cases} \min_{(\xi,\mathbf{f})} & \widehat{G_\alpha(\xi,\mathbf{f})} \equiv \xi + \dfrac{1}{\alpha N}\displaystyle\sum_{i=1}^{N}\left[\left(x_i - f_i + \widehat{\Pi(\mathbf{f})} - \xi\right)^+\right], \\ \text{s.t.} & 0 \le f_i \le x_i,\ \ i = 1, 2, \cdots, N,\ \ \widehat{\Pi(\mathbf{f})} \le \pi. \end{cases}$$

- Empirical CTE minimization model:

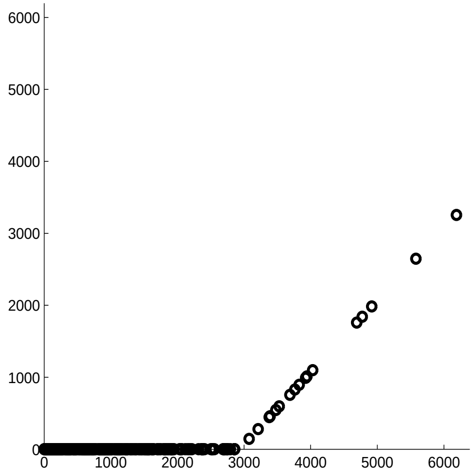
$$\begin{cases} \min_{(\xi,\mathbf{f},\mathbf{z})} & \xi + \dfrac{1}{\alpha N}\displaystyle\sum_{i=1}^{N} z_i, \\ \text{s.t.} & 0 \le f_i \le x_i,\ \ \widehat{\Pi(\mathbf{f})} \le \pi, \\ & z_i \ge 0,\ \ z_i \ge \widehat{\Pi(\mathbf{f})} - f_i - \xi + x_i, \\ & i = 1, 2, \cdots, N. \end{cases}$$

- The above empirical model can be cast as Second-order conic programming for as many as ten reinsurance premium principles.

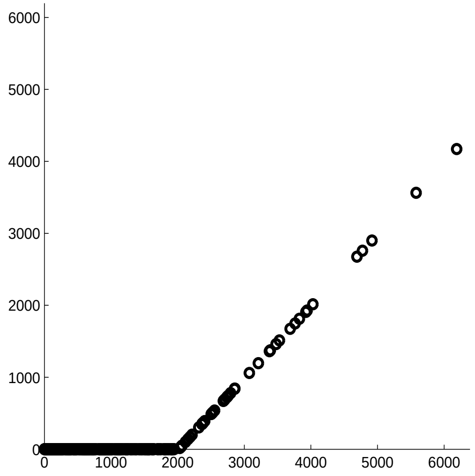
Example: CTE minimization

- Use samples from an exponential loss distribution with mean 1000
- Reinsurance premium principle:
  - expectation principle with loading factor $\theta = 0.2$
  - standard deviation principle with loading factor $\beta = 0.2$
- Consider different levels of the reinsurance premium budget
- The solutions $\mathbf{f}^*$ are illustrated by their scatter plots against sample $\mathbf{x}$

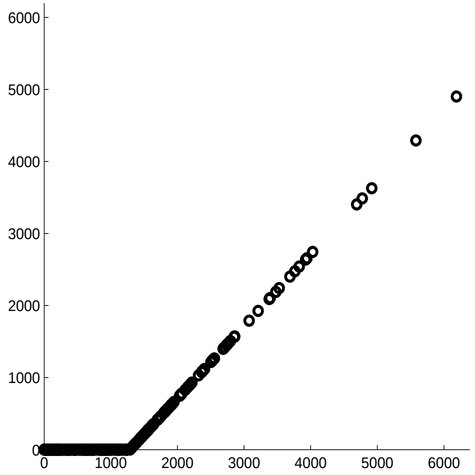
Soln's: CTE min & Expectation Principle (1/2)



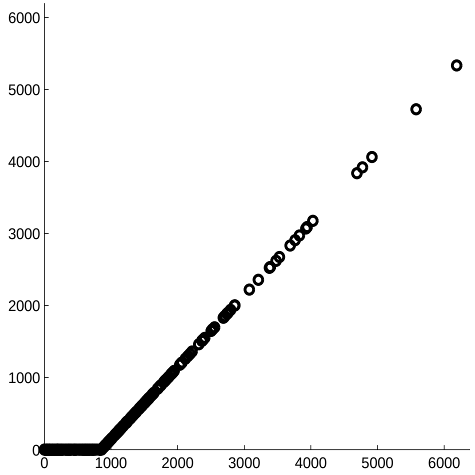
1) $\pi = 80$



2) $\pi = 200$

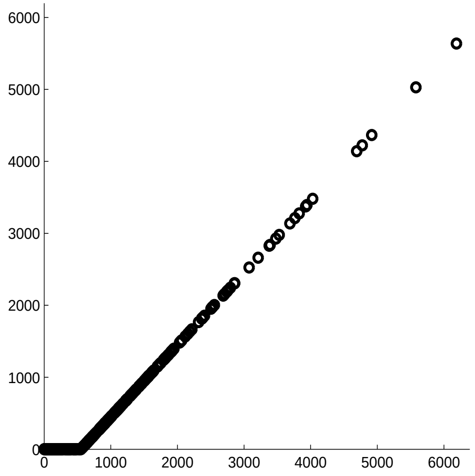


3) $\pi = 400$

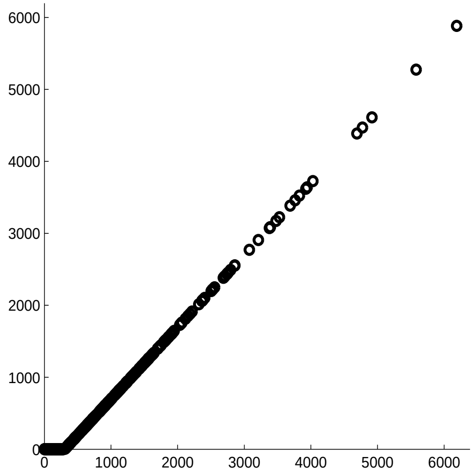


4) $\pi = 600$

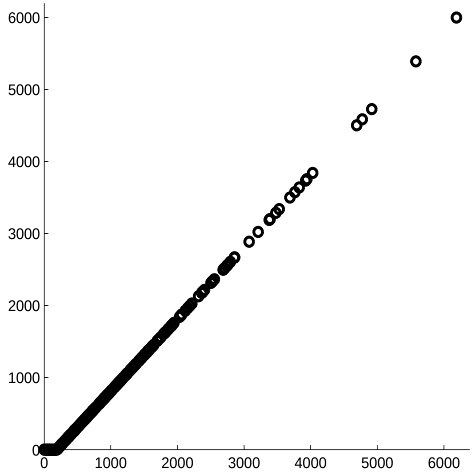
Soln's: CTE min & Expectation Principle (2/2)



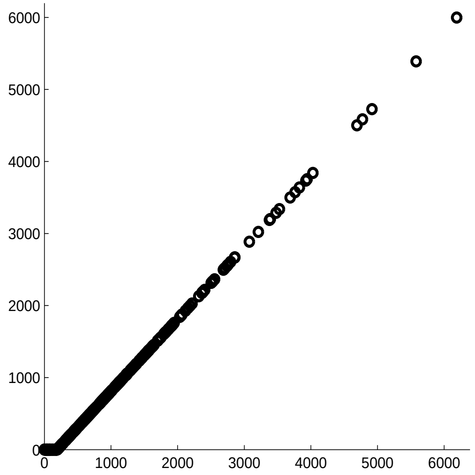
5) $\pi = 800$



6) $\pi = 1000$

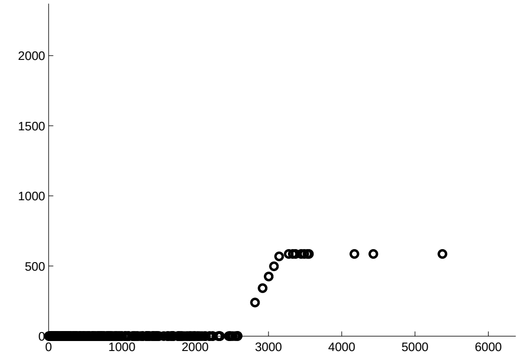


7) $\pi = 1500$

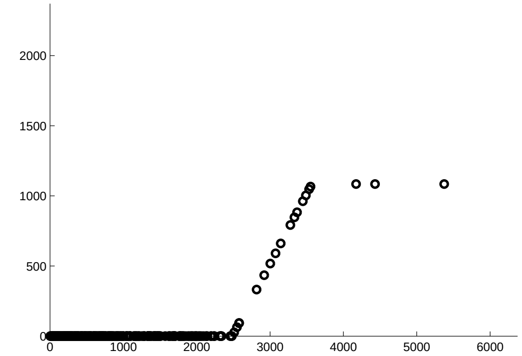


8) $\pi = 2000$

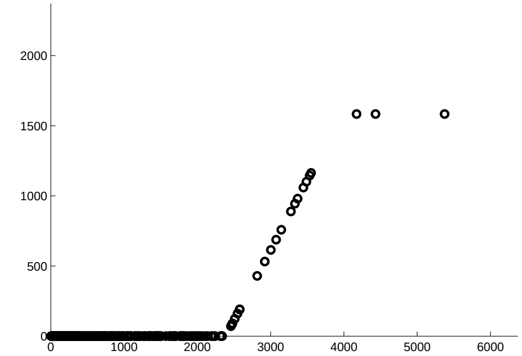
Soln's:: CTE min & Std Principle (1/2)



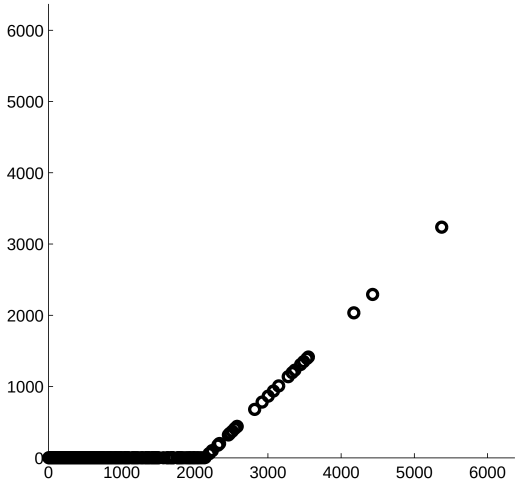
1) $\pi = 50$



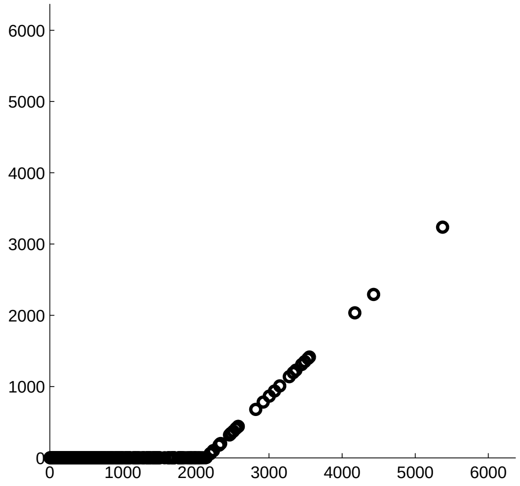
2) $\pi = 80$



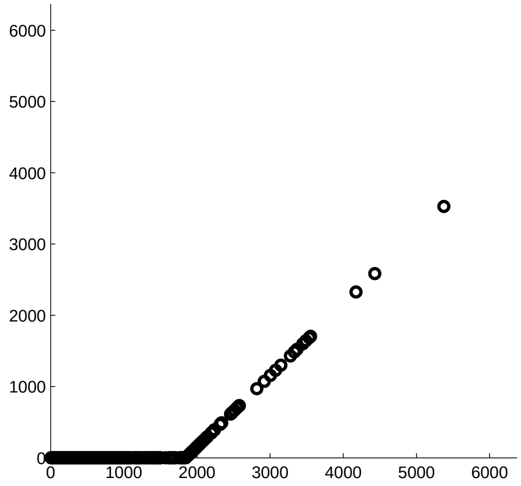
3) $\pi = 100$



4) $\pi = 120$

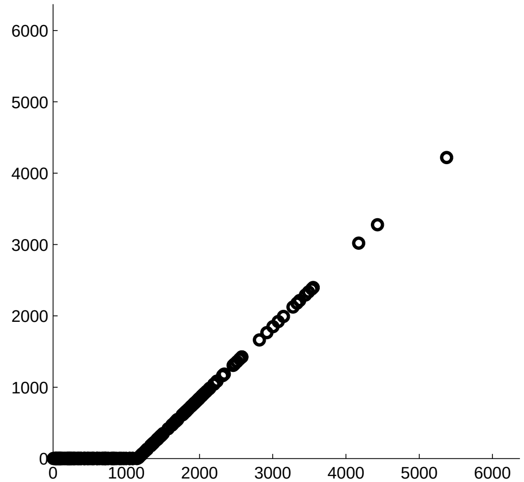


5) $\pi = 150$

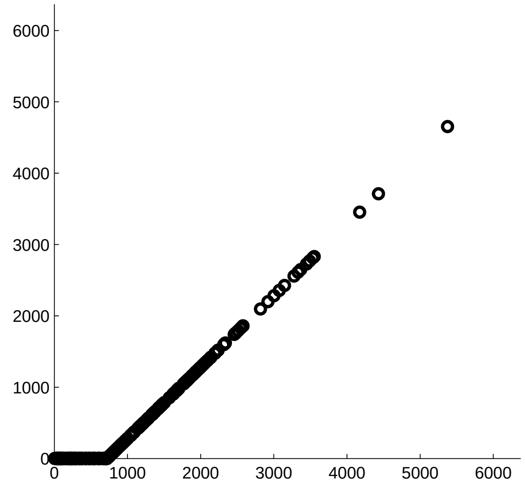


6) $\pi = 200$

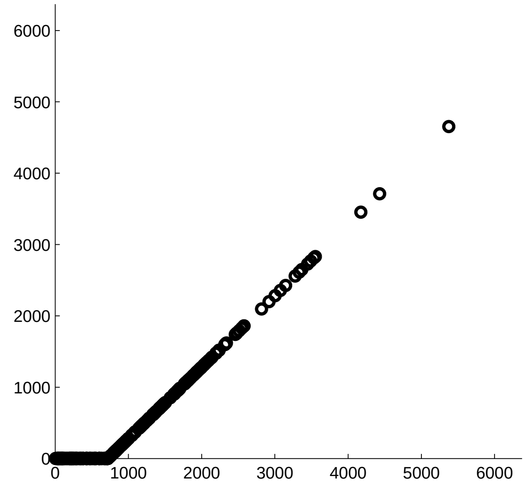
EA: CTE min & Std Dev Principle (2/2)



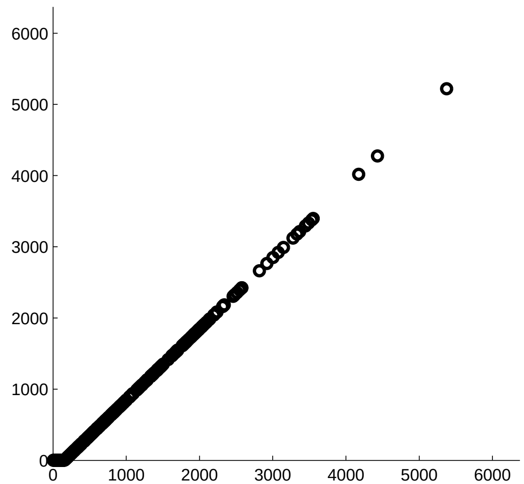
7) $\pi = 400$



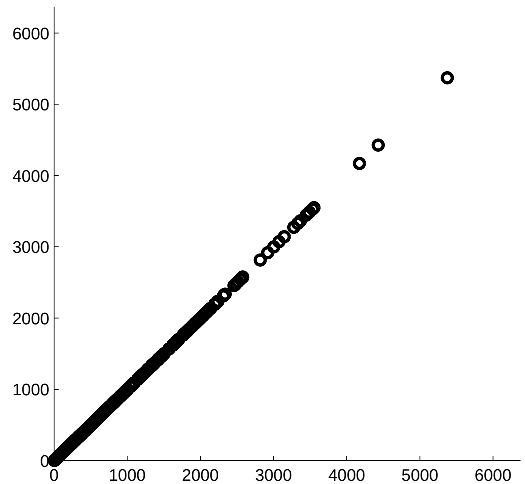
8) $\pi = 600$



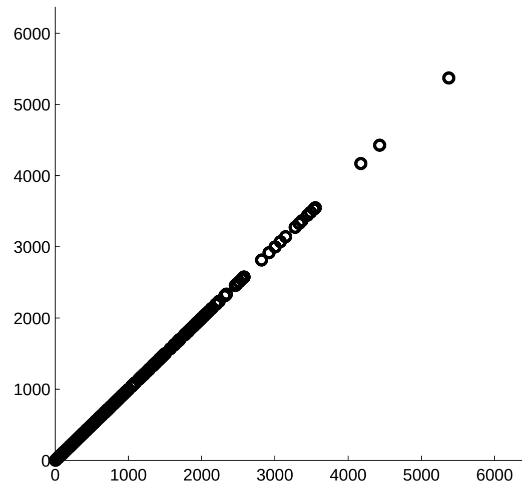
9) $\pi = 800$



10) $\pi = 1000$



11) $\pi = 1500$



12) $\pi = 1500$

Conclusion: Pros & Cons

- **Pros:**
  - empirical data based
  - finite dimensional reinsurance models
  - flexibility of the goal function and the reinsurance premium principle
  - empirical solutions are consistent with the theoretical solutions
    - e.g., Variance and CTE minimization with expectation premium principle
- **Cons:**
  - empirical-based model will turn out to be a large scale programming when the sample size is extremely large

$\Longrightarrow$ issues such as computational time and requirement for a substantially large computer's memory will arise

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