Analytical formulas for Variable Annuity Guarantees and applications, with focus on Stochastic-on-stochastic valuations of hedged Variable Annuity liabilities

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Abstract: *We derive analytical formulas for the fair market values of the guarantee cost and allocated guarantee fee for variable annuities with dynamic lapses, which have important practical applications for stochastic-on-stochastic (SOS) valuations. A comparison of using the analytical formulas derived to the SOS Monte Carlo method estimates a potential reduction in runtime of the order of $1E-07$. The approach derives Black and Scholes (B&S) equations, proposes appropriate boundary conditions and analytically solves the equations using eigenfunction expansion techniques. Analytically solving B&S equations with dynamic lapses, which is important to assume in practice, has not received much attention and numerical algorithms are instead often used to derive the solution. For educational purposes practical actuarial concepts, such as hedge target, SOS valuations and dynamic lapses, are also discussed. The latter are translated in more mathematical terms to derive the B&S equations. For completeness the analytical formulas are also numerically implemented, where we take the opportunity to illustrate the increased cost of guarantees (in percentage terms) due to more efficient policyholder behavior as modeled by dynamic lapses. To simplify the presentation we focus on plain Guaranteed Minimum Maturity Benefits (no resets ect…), but other guarantees will more briefly be discussed. This paper adopts assumptions consistent with how policy liabilities (reserves) are calculated for hedged variable annuities by actuarial practitioners, such as dynamic lapses and mapping funds across multiple market indices, and has the objective of offering practical solutions in particular for pricing and valuation of policy liabilities.*

1. Introduction:

The purpose of this article is to propose analytical formulas for the fair market value of the guarantee cost, denoted $G(.)$, and the fair market value of the allocated guarantee fee, denoted $F(.)$, for variable annuity products in the presence of dynamic lapses. Deriving such analytical formulas has important practical applications, such as in the pricing, risk management and in particular in addressing the numerical challenges of stochastic-on-stochastic (SOS) valuations (also called Nested calculations).

Use of analytical formulas might for instance allow actuarial software providers to offer alternatives to SOS calculations to more efficiently handle valuations of hedged variable annuities. As we will see, explicit analytical calculations can be carried out (and implemented in spreadsheets tools) as illustrated in sections 8 and 9, when the dynamic lapse rates are constant across a number of moneyness levels, which is reasonable to assume in the context of liability modeling. This might also, for instance, allow pricing and model vetting actuaries to build independent tools to validate the cost of guarantee calculations, which usually rely on complex stochastic models. Note that as discussed in section 10, a comparison of using the analytical formulas derived to the SOS Monte Carlo method estimates a potential reduction in runtime of the order of $1\mathrm{E}-07$.

Variable Annuity products, also called Segregated Funds in Canada, offer different guarantees and product features and have been extensively documented in the literature, see [11]. These are in essence investment vehicles, similar to mutual funds, which in addition offer minimum investment guarantees. They are valuable tools for retirement and financial planning, especially in the absence of strong government or corporate sponsored retirement plans. Total assets under management for these products have surpassed 2 Trillion in North America, with annual sales continuing in the magnitude of well over a 100 Billion a year. Variable Annuities are often dynamically hedged since they expose product writers to significant market risk. Companies with a growing block of hedged variable annuities face an increasing computational demand, especially from SOS valuations.

Guaranteed Minimum Maturity Benefits (GMMB) is one of the popular guarantees offered, which guarantees that at maturity T the account value does not fall below a certain guarantee amount K. To better illustrate the mathematical approach, we limit the scope in this article to plain GMMBs (without resets and rollup features etc…), which generally account for a large share of the variable annuities market as pointed out in [10]. However, similar mathematical techniques apply for other guarantees, such as Guaranteed Minimum Death Benefits (GMDB), Guaranteed Minimum Withdrawal Benefits (GMWB) and Guaranteed Life Withdrawal Benefits (GLWB), which for completeness will be briefly discussed later in the article.

Our approach to determine analytical formulas is to derive partial differential equations for both $G(.)$ and $F(.)$ (that we also refer to as Black and Scholes equations (B&S)), propose appropriate boundary conditions and present a general and intuitive mathematical method for deriving the solutions based on the eigenfunction expansion method. This method has been used in mathematical finance to estimate the price of options, but has not been given attention in actuarial practice. In deriving the B&S equations we assume a risk neutral, multivariate lognormal market model with deterministic volatilities and interest rates. In our applications to SOS valuations we are assuming that such a market model is used in calculating the economic hedging measure $H(.) \stackrel{\text{def}}{=} G(.) - F(.)$ (also called hedging target) in the SOS inner-loop calculations. The SOS outer-loop calculations might use other market models, such as the Regime Switching Lognormal models which better captures the fatter tail of the market returns distribution as documented in [11]. More details on SOS valuations, also discussed in [2], [5] and [14], are provided in section 2, where we will in particular explain how analytical expressions of $H(.)$ readily provide the "Greeks – such as Deltas" required to model hedging cash flows in liability reserving models, thus offering an alternative to the inner-loop risk neutral stochastic calculations.

The article intends to educate and be comprehensive as it also offers more details on practical actuarial concepts, such as hedge target, SOS valuations and dynamic lapses. It documents how these concepts translate in a more mathematical framework, from which B&S equations directly follow thanks to the Feynman-Kac formula. Further assuming the fund to follow a fixed allocation investment strategy across market indices (commonly referred to in practice as fund mapping) greatly simplifies the mathematical problem as the B&S equations will only depend on one space variable (as opposed to multiple variables as studied in [16]). Given the focus on actuarial applications the article makes the effort to avoid when possible presenting more theoretical mathematical proofs. The theoretical study is complemented at the end by a numerical implementation of the analytical formulas derived.

We assume the policyholder to behave more efficiently, lapsing more when the moneyness level as represented by the ratio $\frac{y}{K}$ is higher: a feature usually referred to as *Dynamic lapsation*, which is important to assume in practice, see [8] for example. Here y and K are, respectively, the account value and guarantee value at time t. We further assume that:

- The fund follows a fixed allocation investment strategy: its returns are expressed as a time-constant linear combination of returns of a set of market indices (such as S&P/TSX, S&P500 and DEX Bond Index).
- The market prices of indices follow a multivariate, lognormal distribution under a risk neutral measure, with deterministic volatility and correlation factors.
- The forward risk-free rates, denoted $\boldsymbol{r}(t)$, are deterministic with respect to the time variable t.

These market assumptions might seem simplistic, but it is important to note that they are only assumed for liability calculation, which usually relies on long term assumptions. Under these assumptions the guarantee cost $G(t, y)$ satisfies the following equation:

$$\frac{\partial G}{\partial t} + (\boldsymbol{r} - \boldsymbol{m})y\frac{\partial G}{\partial y} + \frac{\boldsymbol{\sigma}^2}{2}y^2\frac{\partial^2 G}{\partial y^2} - \left(\boldsymbol{r} + \boldsymbol{q}\left(t, \frac{y}{K}\right)\right)G = 0, y > 0, t \in (0, T), \text{ with } G(T, y) = Max(0, K - y), \tag{1}$$

which is similar to the usual B&S equation, with the added decrement factor $\boldsymbol{q}$**,** which is sum of the lapse rate and the mortality rate. Here $\boldsymbol{\sigma}$ is a volatility parameter, function of the volatilities and correlation factors of the market indices considered in the fund mapping. The parameter $\boldsymbol{m}$, which we assume to be constant, is the Marginal Expense Ratio (MER) representing in practice the rate of daily fund premium (also called fee) deductions to cover for the guarantees cost, expenses, profit margins and other charges. Likewise, the fair market value of the allocated guarantee fees $F(t, y)$ satisfies a similar, but inhomogeneous equation, as follows:

$$\frac{\partial F}{\partial t} + (\boldsymbol{r} - \boldsymbol{m})y\frac{\partial F}{\partial y} + \frac{\boldsymbol{\sigma}^2}{2}y^2\frac{\partial^2 F}{\partial y^2} - \left(\boldsymbol{r} + \boldsymbol{q}\left(t, \frac{y}{K}\right)\right)F = f\, y, \qquad y > 0, t \in (0, T), \text{ with } F(T, y) = 0. \tag{2}$$

Here f is the fee rate allocated to cover (fully or partially) for the guarantee cost, which is hedge as reflected by the definition $H(.) \stackrel{\text{def}}{=} G(.) - F(.)$ of the hedged target (sometimes f is also said to be bifurcated from $\boldsymbol{m}$). The Equations (1) and (2), with the final conditions specified at $t = T$, fully and uniquely determine $G(.)$ and $F(.)$.

B&S equations have already been used, such as in [4], [13], [12] and [14], in calculating the guarantee cost of variable annuities, but often numeric methods are used and analytic solutions are only derived assuming non-dynamic lapse rates. The latter is an easier problem to solve since, as shown in [16] in a more general setting when the coefficients $\boldsymbol{\sigma}$, $\boldsymbol{r}$, $\boldsymbol{m}$ and $\boldsymbol{q}$ only depend on t (i.e. not on y), the B&S equation can be transformed to one with time-constant coefficients. Deriving exact closed form solutions is not possible with dynamic lapses in the most general case, and the purpose of this article to be precise is to derive approximate analytic formulas for the solution. In [17] the authors determine analytical approximations of Green functions, which are not always practical to implement to derive the solution when they require numerical integrations. In [3] the authors assumed $\boldsymbol{q} = 0$, with $\boldsymbol{\sigma}$**,** $\boldsymbol{r}$ and $\boldsymbol{m}$ not depending on t, and used eigenfunction expansion techniques to derive analytical expressions of the guarantee cost for barrier options (which allowed assuming zero Dirichlet boundary condition). The approach we propose is based on eigenfunction expansion techniques, by restricting equations (1) and (2) on a bounded domain $[\frac{1}{Y}, Y]$, for a certain large enough $Y > 0$, but below are some important differences:

- The approach proposes use of combinations of Dirichlet type and Neumann type boundary conditions, derived as shown in section 4 from the asymptotic behavior of G and F, as $y \to 0$, and as $y \to +\infty$. These boundary conditions, which have not been well documented in the literature, are also important to be specified if we want to solve the B&S equations numerically.
- We solve in addition for the fair value of the allocated guarantee fees, which has usually not received as much attention.
- Proposes a recursive formula to account for the case where the coefficients $\boldsymbol{\sigma}$**,** $\boldsymbol{r}$ and $\boldsymbol{q}$ are time dependent.

As we will see the Neumann type boundary conditions are homogeneous (i.e. the derivative is imposed a zero value), which has the advantage of providing simpler approximation expressions for G and F. These conditions may not always apply, so for completeness we also provide in the appendix expressions based on selecting non-zero Dirichlet type boundary conditions (i.e. the solution is imposed a non-zero value).

The approach we propose is composed of three steps of increased complexity as follows:

1/ The coefficients $\boldsymbol{\sigma}$, $\boldsymbol{r}$ and $\boldsymbol{q}$ are independent of $\boldsymbol{t}$: Depending on the boundary conditions selected, we provide eigenfunction expansions for the solution. We get a simpler formula when selecting the zero Neumann boundary conditions, at $y = Y$ and $= \frac{1}{Y}$, as follows:

$$G(t,y) \approx \sum_{i=1}^{R} c_i\, w_i(\ln(y))\, e^{-\lambda_i(T-t)}, \ \text{ with } \ c_i = \int_{-\ln(Y)}^{+\ln(Y)} Max(0, K - e^x) w_i(x) e^{-\left(1-2\frac{\boldsymbol{r}-\boldsymbol{m}}{\boldsymbol{\sigma}^2}\right)x} dx,$$

with convergence as $Y, R \to +\infty$. The formula for $F(t,y)$ is as follows:

$$F(t,y) \approx \sum_{i=1}^{R} b_i\, y\, \phi_i(\ln(y))\, \frac{1 - e^{-\omega_i(T-t)}}{\omega_i}, \ \text{ with } \ b_i = \int_{-\ln(Y)}^{+\ln(Y)} f\, \phi_i(x) e^{\left(1+2\frac{\boldsymbol{r}-\boldsymbol{m}}{\boldsymbol{\sigma}^2}\right)x} dx.$$

Here w_i (resp. ϕ_i) $i \geq 1$, are eigenfunctions of a Sturm Liouville problem, and λ_i $i \geq 1$, (resp. ω_i) the corresponding eigenvalues. Determination of the eigenfunctions and eigenvalues of Sturm Liouville problems have been extensively studied, see [18]. It is important to note that when the dynamic lapse rates are constant across a number of moneyness levels, which is reasonable to assume in practice, w_i and ϕ_i are easily determined with explicitly and exact analytic expressions.

2/ $\boldsymbol{\sigma}$, $\boldsymbol{r}$ and $\boldsymbol{q}$ are stepwise constant with respect to $\boldsymbol{t}$: A recursive formula is proposed to express the solutions in terms of eigenfunction expansions. It is important to note that when $\boldsymbol{\sigma}$ and $\boldsymbol{q}$ do not depend on t the eigenvalues λ_i and ω_i do not need to be numerically recalculated at each iteration in the recursive formula.

3/ The general case and convergence: When $\boldsymbol{\sigma}$, $\boldsymbol{r}$ and $\boldsymbol{q}$ are stepwise continuous with respect to t they can be approximated by stepwise constant functions for a given subdivision of the interval $[0.T]$, for which we can determine analytic formulas following 2/ above. A result established in [16] for example can be used to prove the convergence to the exact solution as the size of the intervals in the subdivision tends to zero.

We propose in 3/ above to approximate with stepwise constant functions derived by averaging $\boldsymbol{\sigma}^2$, $\boldsymbol{r}$ and $\boldsymbol{q}$ on each subinterval in the subdivision considered. This corresponds to the first order term in the Magnus expansion as presented in [15], which presumably improves the convergence. Using higher order terms would provide a faster convergence but is not within the scope of this article. It is a legitimate question whether the recursive formula will outperform the SOS Monte Carlo calculations. As discussed in section 10 however, a comparison of using the analytical formulas derived to the SOS Monte Carlo method estimates a potential reduction in runtime of the order of $1\text{E} - 07$. This is mainly because the recursive analytical formula does not need to be recalculated for different scenarios, and different policies (of same maturity dates for example).

This paper is organized as follows. In sections 2, we discuss the numerical challenges of SOS valuations and explain how they could be addressed by deriving analytic approximations for G and F. In section 3 we discuss the dynamic lapses and express expected guarantee cost and guarantee fees in more mathematical terms. In section 4 we derive the B&S equations (1) and (2). In section 5 we determine asymptotic behavior of both G and F, as $y \to 0$, and as $y \to +\infty$, from which appropriate boundary conditions are derived when restricting equations (1) and (2) on a bounded domain $y \in [\frac{1}{Y}, Y]$. In section 6 we derive analytic expressions when $\boldsymbol{\sigma}$, $\boldsymbol{r}$ and $\boldsymbol{q}$ are independent of t. Section 7 presents a recursive formula for when $\boldsymbol{\sigma}$, $\boldsymbol{r}$ and $\boldsymbol{q}$ depend on t. Section 8 presents numerical results and comparisons against exact solutions to show convergence. Section 9 establishes explicit analytic calculations when $\boldsymbol{q}$ is stepwise constant with respect to y, and presents numerical illustrations of the derived analytical formulas. In particular we take the opportunity to illustrate the increased cost of guarantees (in percentage terms) due to more efficient policyholder behavior as modeled by dynamic lapses. In section 10 we compare the analytical recursive formulas derived to the SOS Monte Carlo method and estimate a potential reduction in runtime of the order of $1\mathrm{E}-07$. For completeness, section 11 discuses GMDB, GMWB and GLWB guarantees and derives B&S equations. The recursive formulas corresponding to non-homogeneous Dirichlet boundary conditions are left for the appendix.

2. Computational challenge of Stochastic-On-Stochastic valuations

This section provides more background on the Stochastic-On-Stochastic (SOS) valuations of Variable Annuity liabilities, discusses the concept of hedge target and shows how determining analytic expressions for G and F can address the related computational challenges of SOS valuations.

Stochastic valuations: Actuarial reserves for Variable Annuities are generally calculated based on Real World stochastic valuations in which liability cash flows (such as fees and claims) are projected over a large number of economic scenarios, denoted here $RW_k, k = 1, \ldots, R$. Actuarial reserves are accounting measures of liabilities, as opposed to economic measures representing fair market values of liabilities. Reserving follows accounting and regulatory rules, such AG43 actuarial guidelines in the US and also CALM reserving method in Canada. The Real World scenarios are generally calibrated based on historical market experience depending on the applicable actuarial standards. As a reminder, a popular market model used to generate Real World scenarios is the Regime Switching Lognormal Model as documented in [11]. On a high level, stochastic valuations follow the following steps:

- For each policy, net liability cash flows (claims less fees) are projected until maturity of the contract.
- Net cash flows are discounted using a valuation interest rate and aggregated over all policies, thus providing a distribution of values: $PV(CF_k), k = 1, \ldots, R$.
- Statistical measures, such as the Conditional Tail Expectation $CTE(l)$ at a certain confidence level l (for example $l = 75\%$), see in [11], are then applied to that distribution to determine the actuarial reserve. A similar process applies in determining the required capital.
- Depending on the actuarial or regulatory standards additional processing, such as flooring the reserve, is applied to determine the final reserve.

The Hedge Target: the notion of Hedge Target is at the center of any dynamic hedging programs and represents the liability measure considered for hedging market risks, usually determined based on a market consistent basis as opposed to accounting basis. An example of Hedge Target is the fair market value of the guarantee cost $G(.)$. However, given that the premiums for variable annuities are collected by continuous MER deductions over the lifetime of the contract, it might be more appropriate to also hedge a portion of the premiums collected, that we call "allocated guarantee fees" in this article. In that case the Hedge Target is defined as the fair market value of the guarantee cost net of allocated future guarantee fees:

$$H(t, y) \overset{\text{def}}{=} G(t, y) - F(t, y), \tag{3}$$

which is then aggregated across all policies in the inforce. Here $F(t, y)$ is the fair market value of future allocated fees at a rate of f, and $y = V(t)$ is the account value at time t. The rate f for the allocated fees can be determined at a policy level, or at an aggregate level.

In dynamic hedging programs sensitivities (also called Greeks) of $H(t, y)$ to different market impacts, depending on the risk being hedged, are calculated. Examples of first order Greeks are:

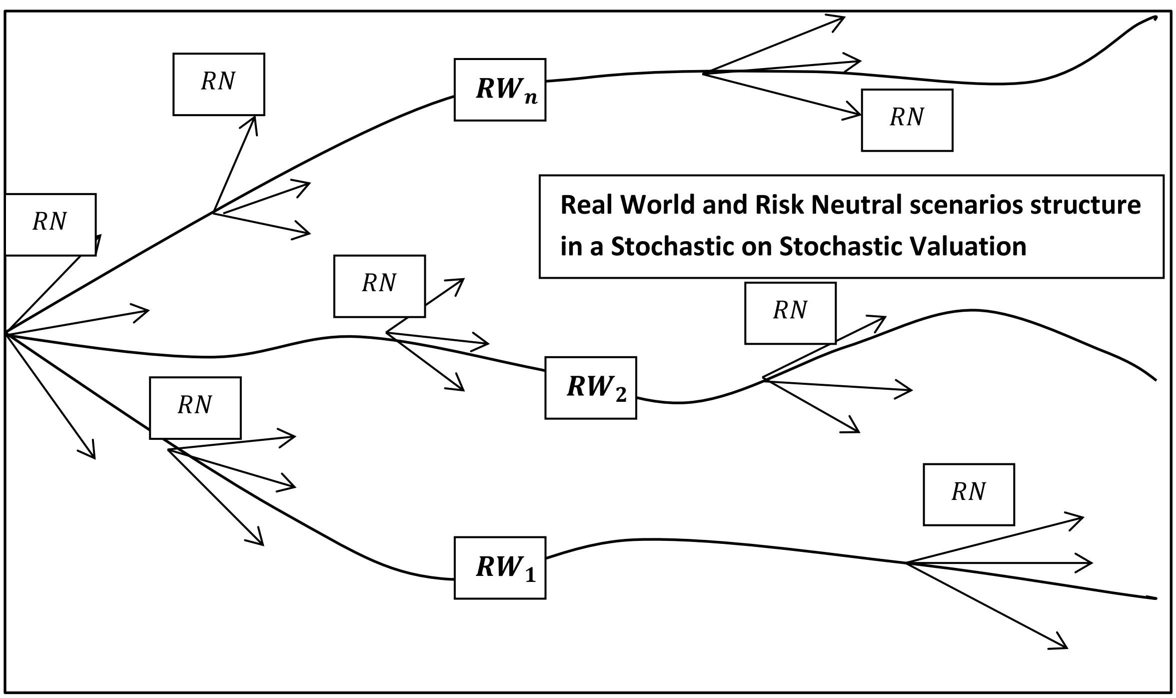
- Delta, which represent the sensitivity of $H(t, \mathrm{y})$ to account value changes, which is mathematically the derivative $\frac{\partial H}{\partial y}$ with respect to y.
- Rho represents the sensitivity to changes in interest rate levels.
- Vega represents the sensitivity to changes in market volatility.

Greeks determine the amount of hedge assets (such as index futures and index options) required for hedging market risks. The asset positions need to be dynamically rebalanced over time, which generates profits and losses, also called hedge cash flows, see [2].

SOS valuations and the computational challenge: The reserve valuation of hedged variable annuities requires projection of the additional streams of cash flows, namely the modeled hedge cash flows as defined above. The latter are then combined with other liability cash flows (claims and fees) in order to determine the reserve as described earlier.

In practice the calculation of the Hedge Target Greeks, required to determine the modeled hedge asset positions, requires additional stochastic Risk Neutral valuations, which gives rise to SOS calculations as illustrated in the graph below. More precisely, for each policy, each Real World scenario $RW_k, k = 1, \dots, R$, and at each projected time point t, Risk Neutral stochastic calculations (denoted by *"RN"* in the graph below) are performed to determine the value of the Hedge Target and its sensitivities to market drivers (i.e. the Greeks). SOS valuations are also known as Nested stochastic valuations. To distinguish between Real World and Risk neutral scenarios, the former are sometimes referred to as outer-loop scenarios, and the latter as inner loop-scenarios.

Chart 1: *Structure of a Stochastic On Stochastic valuation. RW scenarios are called outer loop scenarios, and RN scenarios are called Inter loop scenarios*.



The additional required risk neutral stochastic calculations add much computational demand on an already computationally extensive stochastic valuation. The large number of combinations of policies (e.g. 1 Million), Real world scenario (e.g. 5000), risk neutral scenarios (e.g. 1000) and points in time (e.g. 600 monthly time-steps for a 50 years projection period) stretch the computation resources to an extreme.

Analytic expression for the hedge target offer an alternative to SOS valuations: Given the usually large number of policies it may not be possible to complete SOS valuations within the required reporting timeline. Approximation techniques used to speed up the runs, as discussed in [2] (see also [5] and [14]), may still be computationally expensive and come with reduced accuracy and granularity for the results.

The ideal method to overcome this computational challenge is to use exact or approximate analytic expression for the hedge target, as mentioned in [2]. The inner-loop runs would then not be needed, as the analytic expressions readily provide the Greeks of the hedge target, which makes the valuation computationally much more manageable. An alternative to determining analytic expressions would also be to numerically solve equations (1) and (2), such as using finite difference methods or finite volume methods. Such an approach might still be computationally expensive unless efficient numerical methods are used. Note here that the boundary conditions, such as those we derive in section 3, are required should numerical methods are to be used.

3. Dynamic lapses and mathematical formulations for expected guarantee cost and fees

This section discusses the dynamic lapse assumption, and provides a mathematical formulation for the expected guarantee claims and expected allocated guarantee fees as functions of the lapse rate, which will be used in the next section to derive B&S equations.

Some notations and definitions: Guaranteed Minimum Maturity Benefits (GMMB) guarantee that at time of maturity T the account value does not fall below a guaranteed value. For the guarantee to apply the policyholder is required to be still alive and to maintain his policy up to maturity. Dying or surrendering the contract before maturity expires the guarantee. Let us denote by $Q(t,s)$ the probability of a policy to "survive" lapses and mortality decrements between t and $s \geq t$ given that it is inforce at time t. Let us denote by $V(s)$ the value of the account at time $s \geq t$, and by $S(s)$ the value of the underlying fund (i.e. the account value ignoring decrements) given that $V(t) = S(t)$. We then have

$$V(s) = Q(t,s)S(s). \tag{4}$$

If we assume that decrements proportionately reduce the projected guarantee value $K(s)$ then we also have:

$$K(s) = Q(t,s)K(t). \tag{5}$$

The next subsection also provide a non-probabilistic interpretation of $Q(t,s)$ so to justify (4) and (5). At time t, the expected guarantee claim for a GMMB with a guarantee value of $K(t) = K$ at time t is then given by:

$$\text{Expected Guarantee Claim} = Max\left(0, Q(t,T)K - V(T)\right) = Q(t,T)Max\left(0, K - S(T)\right). \tag{6}$$

This expected cost is contingent on the realized market scenario between t and T. In practice the Guarantee value is usually elected at inception of the contract as a percentage (e.g. 75% or 100%) of the initial deposit. In the absence of decrements the guarantee value stays constant, unless the product offers features such as resets. Higher decrements result in lower expected claims in dollar terms as reflected by formula (6).

On a daily basis the variable annuity writer deducts premiums at a fixed rate m of the fund, usually called the Marginal Expense Ratio (MER), to cover for the cost of the guarantee expenses and other charges, while allowing for a profit margin. In a context of pricing, the cost of the guarantee for a given policy maybe measured as a portion $f' < m$ of the MER, and determined at inception of the policy so that the Fair Market Value of total expected fees at the rate f' is equal to the Fair Market Value of total expected claims. In the context of hedging the allocated guarantee fee f to be hedged, as reflected in the definition of the hedge target in (3), maybe defined to be equal to f', or higher (resp. lower) if we want to hedge a higher (resp. lower) portion of the MER. In practice, the hedged fee f maybe determined on a cohort level, and not necessarily at the policy level.

Mathematically, the infinitesimal allocated fee charged at the rate f in the time interval $[s, s+ds]$ is $f\,V(s)ds$. The total expected guarantee fees charged between time t and T at a rate f is then given by:

$$\text{Expected Allocated Guarantee Fee} = \int_t^T f\,V(s)ds = f\int_t^T Q(t,s)S(s)ds. \tag{7}$$

Expected Guarantee claims and expected guarantee fees as functions of dynamic lapse rates: Let $\boldsymbol{q}(s)$ be the average rate of decrements at time $s \geq t$ so that $d_s Q(t,s) = -\boldsymbol{q}(s)Q(t,s)ds$. Here $\boldsymbol{q}(s) = \boldsymbol{q}_w(s) + \boldsymbol{q}_d(s)$, where $\boldsymbol{q}_w(s)$ is the force of decrements due to lapses, also called lapse rate in actuarial modeling terminology. $\boldsymbol{q}_d(s)$ is the rate of decrements due to mortality, also called mortality rate, or force of mortality.

The mortality rate depends on the attained age at time s, which we will not indicate for ease of the presentation. We can view the lapse rate as an average, accounting for both full and partial surrenders of fund units. It is important to note that for most GMMB product designs, which we assume here, partial surrenders reduce the guarantee value proportionately. The probability $Q(t,s)$ introduced above could then be viewed as the proportion of units of a policy that stay inforce up to time s as compared to the number of units at time t, which supports formula (6).

By definition, the moneyess level at a future point in time $s \geq t$ is the ratio of the account value to the guarantee value: $\frac{V(s)}{Q(t,s)K}$, which is equal to $\frac{S(s)}{K}$ using (4) and (5), where $K = K(t)$ is the guarantee value at time t. As the moneyness increases (resp. decreases) the guarantee will be perceived less valuable (more valuable), resulting in higher lapses (lower lapses) for an efficient policyholder. This fact is actually confirmed by experience and reflected in many actuarial models for pricing, valuation and dynamic hedging. The lapse rate $\boldsymbol{q}_w\left(s, \frac{V(s)}{Q(t,s)K}\right) = \boldsymbol{q}_w\left(s, \frac{S(s)}{K}\right)$ then depends on both the time variable s and moneyness level. The dependence on the moneyness level is often referred to by saying that the lapse rate is dynamic. This is a distinctive feature of variable annuity products, adding complexity to the valuation of these products and makes them more costly. We can also refer to [8] for a discussion on dynamic lapses.

Note that since $\boldsymbol{d}_s Q(t,s) = -\boldsymbol{q}\left(s, \frac{S(s)}{K}\right) Q(t,s)ds$ we have

$$Q(t,T) = Q(t,t)e^{-\int_t^T \boldsymbol{q}\left(s,\frac{S(s)}{K}\right)ds}, \text{ where } Q(t,t) = 1.$$

From formulas (6) and (7) we then have the following mathematical formulation for the expected guarantee claims and expected guarantee fees as functions of lapse rates.

Definition: *We denote by $G(t,y)$ the Fair Market Value at time $t \leq T$ of the GMMB expected guarantee claim, given that the account value $V(t) = y$ and the guarantee value $K(t) = K$. The expected guarantee claim, for a given realized market scenario between t and s, is equal to:*

$$Max\left(0, e^{-\int_t^T \boldsymbol{q}\left(s,\frac{S(s)}{K}\right)ds}K - V(T)\right) = e^{-\int_t^T \boldsymbol{q}\left(s,\frac{S(s)}{K}\right)ds} Max\left(0, K - S(T)\right). \tag{8}$$

For simplicity, the dependence of $G(t,y)$ with respect to K and T is omitted in the notation, unless needed.

Likewise, we denote by $F(t,y)$ the Fair Market Value at time $t \leq T$ of the expected allocated guarantee fee charged at the rate f between t and T. The expected value is given by the following mathematical formula:

$$\int_t^T e^{-\int_t^s \boldsymbol{q}\left(p,\frac{S(p)}{K(p)}\right)dp} fS(s)ds. \tag{9}$$

The Fair Market Values $G(t,y)$ and $F(t,y)$ are calculated in practice using Stochastic Monte-Carlo Risk Neutral calculations, as will be discussed in the next section. In a pricing framework, if the guarantee fee rate f is determined at policy issue t_0 so that

$$F(t_0,y) = fF_{f=1}(t_0,y) = G(t_0,y), \quad \text{which implies} \quad f = \frac{G(t_0,y)}{F_{f=1}(t_0,y)}, \tag{10}$$

then f is a measure of the "price" of the guarantee as a percentage of the projected account values. In this case, a higher rate f means a higher guarantee price, resulting in a lower margin $m - f$ to cover for expenses, profit margins and other charges from a pricing perspective.

4. The Risk Neutral valuation and derivation of Black and Scholes equations

We adopt assumptions consistent with those usually used in practice in actuarial models for SOS valuations of hedge variable annuities. In particular we assume that the hedge target Greeks in the SOS inner-loop runs are calculated based on a risk neutral, multivariate lognormal market model with deterministic volatilities and interest rates. These market assumptions might seem simplistic, but it is important to note that they are only assumed for liability calculation. This section starts by defining the market model in more details, then provides a mathematical formulation for the risk neutral values for $G(.)$ and $F(.)$, from which the Black and Scholes equations directly follow.

<u>Risk neutral formulations for $G(.)$ and $F(.)$</u>: Assume the market to be affected by k sources of randomness, where there is available k tradable assets with prices at time t denoted by $I_i(t), i = 1, \dots, k$. We assume the latter to follow a multivariate lognormal stochastic differential process under a risk-neutral measure $\mathbb{Q}$, with deterministic interest rates and volatilities:

$$dI_i(t) = \boldsymbol{r}(t)I_i dt + \boldsymbol{\sigma}_i(t)I_i dW_i, \;\; i = 1, \dots, k,$$

were $\boldsymbol{r}(t)$ is the forward short term risk free rate. Here $\boldsymbol{\sigma}_i(t)$ is the volatility parameter of I_i(t), and $W_i(t)$, $i = 1, \dots, k$, are correlated Brownian motions under the risk-neutral measure $\mathbb{Q}$ (see [**9**], Chapter 6, on multi-asset models). For each i,j, $\boldsymbol{\rho}_{i,j}(t)$ are the correlation parameters between $W_i(t)$ and $W_j(t)$, expressed in the usual shorthand notation as: $dW_i dW_j = \boldsymbol{\rho}_{ij}dt, \forall i,j$, with $\boldsymbol{\rho}_{i,i} = 1$. Even though lognormal models are not generally adopted in the SOS outer-loop stochastic runs, they are still used in dynamic hedging, as pointed out in [11]. We will also assume the following uniform elliptic condition:

$$\sum_{i,j \leq k} \boldsymbol{\rho}_{i,j}(t)\boldsymbol{\sigma}_i(t)\boldsymbol{\sigma}_j(t)\xi_i\xi_j \geq c \sum_{i \leq k} \xi_i^2, \qquad \forall\, \xi \in\, R^k \text{ and } \forall t \in [0,T]. \tag{11}$$

We can for example think of tradable assets $I_i(t), i = 1, \dots, k$, as traded market indices tracking the performance of different asset classes (e.g. S&P/TSX, S&P500 and DEX Bond Index). In practice, for the purpose of liability modeling, the fund is "mapped" across multiple market indices. This reflects the fact that the fund, with total value $S(t) = S_1(t) + \cdots + S_k(t)$, typically invests in assets $I_i(t), i = 1, \dots, k$, according to a certain investment strategy, where $S_i(t)$ denotes the value of the fund invested in the asset $I_i(t)$. Given that the fund is subject to continuous MER deductions, at a rate of $\boldsymbol{m}$, the following stochastic differential equations hold under the risk-neutral measure $\mathbb{Q}$:

$$dS_i(t) = (\boldsymbol{r}(t) - \boldsymbol{m})S_i dt + \boldsymbol{\sigma}_i(t)S_i dW_i, \quad \text{for} \quad i = 1, \ldots, k.$$

From (8) we can then explicitly express $G(.)$ in terms of the following Risk Neutral valuation formula:

$$G(t, y_1, y_2, \ldots, y_k) = \mathop{\mathbb{E}_{\mathbb{Q}}}_{S_i(t)=y, \forall i} \left(e^{-\int_t^T \boldsymbol{r}(s)ds} \; e^{-\int_t^T \boldsymbol{q}\left(s, \frac{S(s)}{K}\right)ds} Max\left(0, K - S(T)\right) \right), \tag{12}$$

where $y_i = S_i(t)$ is the value of the account at time t invested in asset I_i, and the total account value is $y = y_1 + \cdots + y_k$ (we assumed the fund and the account to have the same values at time t, but no necessarily at $s > t$ due to the effect of decrements). This expression is in practice the basis for the Monte-Carlo risk neutral stochastic calculations. Likewise, the fair market value of the expected future fees (9) is given by:

$$F(t, y_1, y_2, \ldots, y_k) = \mathop{\mathbb{E}_{\mathbb{Q}}}_{S_i(t)=y, \forall i} \left(\int_t^T e^{-\int_t^s \boldsymbol{r}(p)dp} \; e^{-\int_t^s \boldsymbol{q}\left(p, \frac{S(p)}{K}\right)dp} f\, S(s) ds \right). \tag{13}$$

The one-dimensional Black and Scholes equations for $G(.)$ and $F(.)$: The expressions above yield multi-space-variables partial differential equations for $G(.)$ and $F(.)$. However, under the assumption that the fund follows a fixed allocation investment strategy we will show that these equations only depend on one space variable. Under this assumption the proportional allocation α_i of the fund to the asset $I_i, i = 1, \ldots, k$, is held constant over time. This is not too restrictive as it applies to a large proportion of funds in the market. It is also reasonable in practice in actuarial models as the fund mapping is usually not modelled to dynamically change starting from the valuation date. Let us introduce the following notations which will be used in the theorem below:

$$\boldsymbol{\sigma}^2(t) \; = \; \sum_{i,j=1}^{k} \boldsymbol{\rho}_{ij} \alpha_i \boldsymbol{\sigma}_i \, \alpha_j \boldsymbol{\sigma}_j\,, \quad \text{where} \quad \alpha_i = \frac{S_i}{S}, \qquad i = 1, \ldots, k, \text{ are constants.} \tag{14}$$

Theorem: *Under the assumption that the fund follows a fixed allocation rebalancing investment strategy, the function $G(t, y)$ as a function of time t, the value y of the account at t, and the guarantee value K at t, satisfies the following partial differential equation with final condition:*

$$\begin{cases} \dfrac{\partial G}{\partial t} + (\boldsymbol{r} - \boldsymbol{m})y\dfrac{\partial G}{\partial y} + \dfrac{1}{2}\boldsymbol{\sigma}^2 y^2 \dfrac{\partial^2 G}{\partial y^2} - \boldsymbol{r}G - \boldsymbol{q}\left(t, \dfrac{y}{K}\right) G = 0, \qquad 0 < t < T, \forall y > 0, \\ G(y, T) = \text{Max}(0, K - y), \;\; \text{as a final condition.} \end{cases} \tag{15}$$

This equation is similar to the Black and Scholes equation except for additional term $\boldsymbol{q}G$ related to the decrement rate. Likewise, $F(t, y)$ satisfies the following inhomogeneous partial differential equation:

$$\begin{cases} \dfrac{\partial F}{\partial t} + (\boldsymbol{r} - \boldsymbol{m})y\dfrac{\partial F}{\partial y} + \dfrac{1}{2}\boldsymbol{\sigma}^2 y^2 \dfrac{\partial^2 F}{\partial y^2} - \boldsymbol{r}F - \boldsymbol{q}\left(t, \dfrac{y}{K}\right) F + fy = 0, \qquad 0 < t < T, \forall y > 0, \\ F(y, T) = 0, \;\; \text{as a final condition.} \end{cases} \tag{16}$$

This equation is similar to the prior one except for the zero final condition and the presence of the fee related term $f\, y$. When needed to show the dependence with respect to K we will use the notations $G(t, y, K)$ and $F(t, y, K)$. Note that these functions can be solved by assuming $K = 1$ since we have the following identities:

$$G(t, y, K) = KG(t, y/K, 1) \;\; and \;\; F(t, y, K) = KF(t, y/K, 1). \tag{17}$$

Proof: First, note that since the fund follows a fixed allocation rebalancing investment strategy, we have

$$dS = \sum_{i=1}^{k} dS_i = (\boldsymbol{r}-\boldsymbol{m})Sdt + S\sum_{i=1}^{k} \alpha_i \boldsymbol{\sigma}_i dW_i = (\boldsymbol{r}-\boldsymbol{m})Sdt + S\boldsymbol{\sigma}\, dW, \text{ with } W \stackrel{\text{def}}{=} \frac{1}{\boldsymbol{\sigma}}\sum_{i=1}^{k} \alpha_i \boldsymbol{\sigma}_i W_i, \tag{18}$$

where $\boldsymbol{\sigma}$ is defined in (14) and W is a Brownian motion since it is a sum of Brownian motions. Function $G(t,y)$ is then function of the one variable y, and expressions (12) and (13), respectively, become:

$$G(t,y) = \mathop{\mathbb{E}_{\mathbb{Q}}}_{S(t)=y,} \left(e^{-\int_t^T \boldsymbol{r}(s)ds} \; e^{-\int_t^T \boldsymbol{q}\left(s,\frac{S(s)}{K}\right)ds} Max\left(0, K - S(T)\right) \right), \tag{19}$$

and

$$F(t,y) = \mathop{\mathbb{E}_{\mathbb{Q}}}_{S(t)=y} \left(\int_t^T e^{-\int_t^s \boldsymbol{r}(p)dp} \; e^{-\int_t^s \boldsymbol{q}\left(p,\frac{S(p)}{K}\right)dp} f\; S(s)ds \right). \tag{20}$$

To derive the partial differential equation (15) we use the version of the Feynman-Kac formula, referred to with the added term “with Killing” in [**1**] theorem 8.2.1. Likewise, the partial differential equation for $F(t,y)$ in (16) is derived using a modified version of the Feynman-Kac formula, sometimes referred to as the “Running Payoff” formula.

Identities in (17) can be directly established by verifying that functions on the right in each of the equalities are also solutions to the same Black and Scholes equations, and satisfying the same final conditions (then use the uniqueness result of the solutions of the Black and Scholes equations). This completes the proof. **Q.E.D**.

5. Derivation of the appropriate boundary conditions and restriction to a bounded domain

In restricting equation (1) (resp. (2)) on a bounded domain $y \in [\epsilon, Y]$, with $\epsilon = \frac{1}{Y}$ and $Y > 0$ large enough, we are required to assign boundary conditions at $y = Y$ and $= \frac{1}{Y}$, which we derive from the asymptotic values of $G(t,y)$ (resp. $F(t,y)$) as $y \to 0$, and as $y \to +\infty$, that we determine in this section. The solution defined on $y \in [\epsilon, Y]$ converges to $G(t,y)$ (resp. $F(t,y)$) as $Y \to +\infty$. To limit the scope we will not provide a poof for that convergence.

<u>Second order parabolic equations, and determination of asymptotic values</u>: It is convenient to transform equations (1) and (2) into second order parabolic equations using the following well-known change of variables:

$$\tilde{u}_c(\tau,x) = G(t,y), \qquad \text{with} \qquad x = \mathrm{Ln}(y) \;\text{ and }\; \tau = T - t \text{ (the time to maturity).} \tag{21}$$

Equation (1) is then equivalent to the following second order parabolic equation with initial condition:

$$\begin{cases} \dfrac{\partial \tilde{u}_c}{\partial \tau} + \mathcal{A}(\tau)\tilde{u}_c = 0, \qquad 0 < \tau < T, \forall x \in \mathbb{R}, \\ \tilde{u}_c(x,0) = \mathrm{MAX}(0, K - e^x), \;\; \text{(initial condition)}, \end{cases} \tag{22}$$

where $\mathcal{A}(\tau)$ is the following second order differential operator:

$$\mathcal{A}(\tau)u = -\frac{\sigma^2(\tau)}{2}\frac{\partial^2 u}{\partial x^2} + \left(\frac{\sigma^2(\tau)}{2} - (r(\tau) - m)\right)\frac{\partial u}{\partial x} + \left(r(\tau) + q(\tau,x)\right)u, \qquad \forall u, \tag{23}$$

which is uniformly elliptic given (11): $\sigma(\tau) > c' > 0, \forall \tau \in (0,T)$. The parameters of the operator are as follows:

$$\sigma(\tau) \stackrel{\text{def}}{=} \boldsymbol{\sigma}(T-\tau), \qquad r(\tau) \stackrel{\text{def}}{=} \boldsymbol{r}(T-\tau) \;\text{ and }\; q(\tau,x) \stackrel{\text{def}}{=} \boldsymbol{q}\left(T-\tau, \frac{e^x}{K}\right), \quad \tau \in [0,T]. \tag{24}$$

Obviously, solving for $\tilde{u}_c(.)$ fully determines $G(.)$. Similarly, using the following change of variable for $F(.)$:

$$e^{x}\tilde{u}_F(\tau,x) = F(t,y), \text{ with } \ x = \mathrm{Ln}(y) \text{ and } \tau = T - t, \tag{25}$$

we have the following inhomogeneous second order parabolic problem with zero initial condition:

$$\begin{cases} \dfrac{\partial \tilde{u}_F}{\partial \tau} + \mathcal{B}(\tau)\tilde{u}_F = f, & 0 < \tau < T, \forall x \in \mathbb{R}, \\ & , \\ \quad \tilde{u}_F(x,0) = 0, \quad \text{(initial condition)}, \end{cases} \tag{26}$$

where $\mathcal{B}(\tau)$ is the following uniformly elliptic second order differential operator:

$$\mathcal{B}(\tau)u = -\frac{\sigma^2(\tau)}{2}\frac{\partial^2 u}{\partial x^2} - \left(\frac{\sigma^2(\tau)}{2} + (r(\tau) - m)\right)\frac{\partial u}{\partial x} + \big(m + q(\tau,x)\big)u, \qquad \forall u. \tag{27}$$

with the zero-Dirichlet boundary conditions. We have the following theorem:

Theorem: *Assume that $\boldsymbol{q}(t,y)$ is independent of y, for y large enough, equal to a certain function $\boldsymbol{q}_+(t)$. And also that $\boldsymbol{q}(t,y)$ is independent of y, for $y > 0$ small enough, equal to a certain function $\boldsymbol{q}_-(t)$. We denote $q_\pm(\tau) = \boldsymbol{q}_\pm(T-t)$. The solution $\tilde{u}_c(\tau,x)$ of (22) then converges asymptotically as follows:*

$$\tilde{u}_c(\tau,-x) \to K e^{-\int_0^\tau (r(s)+q_-(s))ds}, \qquad \tilde{u}_c(\tau,x) \to 0, \qquad \frac{\partial \tilde{u}_c}{\partial x}(\tau,\pm x) \to 0, \quad \text{as} \quad x \to +\infty. \tag{28}$$

Similarly, the solution $\tilde{u}_F(\tau,x)$ to (26) converges asymptotically as follows:

$$\tilde{u}_F(\tau,\pm x) \to f\int_0^\tau \ e^{-\int_s^\tau q_\pm(p)dp} e^{-m(\tau-s)} ds\,, \qquad \frac{\partial \tilde{u}_F}{\partial x}(\tau,\pm x) \to 0, \quad \text{as} \quad x \to +\infty. \tag{29}$$

Before we proceed with the proof let us note, without giving more details, that the conditions on $\boldsymbol{q}(t,y)$ in the theorem above could be relaxed where we only require a certain convergence of $\boldsymbol{q}(t,y)$ and $\frac{\partial \boldsymbol{q}(t,y)}{\partial y}$ as $y \to +\infty$, and also as $y \to 0$.

Proof: To limit the scope we avoid more theoretical proofs as in [16], and instead only present a heuristic argument based on the risk neutral representations (19) and (20). We refer to [6] for a proof when $\boldsymbol{q} = 0$.

Note that as $y \to 0$, future fund values $S(s), s > t$, given $S(t) = y$, will also converge to zero in a certain sense. Replacing $S(s)$ and $S(T)$ by zero in (19) suggest the following limit value for $G(t,y)$:

$$G(t,y) \to K e^{-\int_t^T (\boldsymbol{r}(s)+\boldsymbol{q}_-(s))ds}, \ \text{ as } \ y \to 0.$$

Using similar arguments we show that $G(t,y)$ converges to 0 as $y \to +\infty$. This determines the limit values for $\tilde{u}_c(\tau,\pm x)$, as $x \to +\infty$, stated in (28), given (21) and (24).

For $F(t,y)$, note first that from (18) the following stochastic process is a martingale

$$s \to S(s) e^{-\int_t^S (\boldsymbol{r}(p)-m)dp},$$

which implies the following when taking the average with respect to the risk neutral measure $\mathbb{Q}$:

$$\underset{S(t)=y}{\mathbb{E}_{\mathbb{Q}}} \ [e^{-\int_t^S \boldsymbol{r}(p)dp} \ \ S(s)] = y e^{-\int_t^S m\,dp} = y e^{-m(s-t)}, \text{ since } m \text{ is constant.}$$

Using similar arguments as for $G(t,y)$, the following expression converges, in some sense, as follows:

$$e^{-\int_t^s \boldsymbol{q}(p,S(p))dp} \to e^{-\int_t^s \boldsymbol{q}_+(p)dp}, \quad \text{as } y \to +\infty.$$

This suggest the following limit value for $F(t,y)$, based on equation (20):

$$F(t,y) \to fy \int_t^T e^{-\int_t^s \boldsymbol{q}_+(p)dp} e^{-m(s-t)} ds \text{ , as } y \to +\infty,$$

and similarly

$$F(t,y) \to fy \int_t^T e^{-\int_t^s \boldsymbol{q}_-(p)dp} e^{-m(s-t)} ds \text{ , as } y \to 0,$$

which establishes the first convergence statements in (29), given (25).

The proof in determining the limit values for $\frac{\partial \tilde{u}_c}{\partial x}(\tau, \pm x)$, as $x \to +\infty$, is out of scope but can be heuristically obtained by taking the derivative with respect to x in (22) so that $\frac{\partial \tilde{u}_c}{\partial x}$ satisfies an initial value second order parabolic problem, the solution of which we already presented a proof to derive the limit values. This completes the proof. **Q.E.D**.

Note that in the case where q_- and q_+ are independent of τ the limits in (29) simplify as:

$$f\left(\frac{1-e^{-(q_+ + m)\tau}}{q_+ + m}\right), \text{ as } x \to +\infty, \text{ and } f\left(\frac{1-e^{-(q_- + m)\tau}}{q_- + m}\right), \text{ as } x \to -\infty.$$

<u>Restriction on a bounded domain, and selection of appropriate boundary conditions:</u> Asymptotic values in (28) suggest four combinations of boundary conditions to consider when restricting equation (22) (respectively, equation (26)) to a bounded domain $[-L, +L]$ for the space variable x. This is equivalent to restricting equations (1) and (2) on the bounded domain $y \in [\frac{1}{Y}, Y]$, with $Y = e^L$. Selecting Dirichlet type boundary conditions at both $x = -L$ and $x = L$ leads to the following second order parabolic equation:

$$\begin{cases} \dfrac{\partial u_c^{\mathcal{D}}}{\partial \tau} + \mathcal{A}(\tau) u_c^{\mathcal{D}} = 0, \qquad 0 < \tau < T, \forall x \in (-L, +L), \; u_c^{\mathcal{D}}(0,x) = \mathrm{Max}(K - e^x, 0), \\ u_c^{\mathcal{D}}(\tau, -L) = K' e^{-\int_0^\tau (r(s) + q_-(s))ds}, \text{ with } \; K' = (K - e^{-L}) \quad \text{ and } \quad u_c^{\mathcal{D}}(\tau, L) = 0. \end{cases} \tag{30}$$

The superscript $\mathcal{D}$ in the notation here is to indicate that the Dirichlet condition is selected at both boundaries. For simplicity the dependence on L is omitted in the notations, unless needed. An alternative to the non-homogeneous Dirichlet condition at $x = -L$ (i.e. non-zero boundary condition) is the homogeneous Neumann condition $\frac{\partial u}{\partial x}(\tau, -L) = 0$. Another alternative is the homogeneous Neumann conditions at both $x = -L$ and $x = L$: $\frac{\partial u}{\partial x}(\tau, -L) = \frac{\partial u}{\partial x}(\tau, L) = 0$. The corresponding function $u_c^{\mathcal{N}}$, where the superscript $\mathcal{N}$ is to indicate that the Neumann condition is selected at both boundaries, is then solution to the following problem.

$$\begin{cases} \dfrac{\partial u_c^{\mathcal{N}}}{\partial \tau} + \mathcal{A}(\tau) u_c^{\mathcal{N}} = 0, \qquad 0 < \tau < T, \forall x \in (-L, +L), \; u_c^{\mathcal{N}}(0,x) = \mathrm{Max}(K - e^x, 0), \\ \dfrac{\partial u_c^{\mathcal{N}}}{\partial x}(\tau, -L) = \dfrac{\partial u_c^{\mathcal{N}}}{\partial x}(\tau, L) = 0. \end{cases} \tag{31}$$

We have the following definition:

Definition: *By definition, the functions* $G^{\mathcal{D}}(t,y)$ *and* $G^{\mathcal{N}}(t,y)$ *defined for* $y \in [\frac{1}{Y}, Y]$, with $Y = e^L$, *are given by*

$$G^{\mathcal{D}}(t,y) \stackrel{\text{def}}{=} u_c^{\mathcal{D}}\big(T-t, \mathrm{Ln}(y)\big), \quad G^{\mathcal{N}}(t,y) \stackrel{\text{def}}{=} u_c^{\mathcal{N}}\big(T-t, \mathrm{Ln}(y)\big), \quad \text{for } y \in [\tfrac{1}{Y}, Y], \tag{32}$$

where $u_c^{\mathcal{D}}$ *and* $u_c^{\mathcal{N}}$ *are defined by* (30) *and* (31) *respectively. We similarly, define functions* $F^{\mathcal{D}}(t,y)$ *and* $F^{\mathcal{N}}(t,y)$ *by*

$$F^{\mathcal{D}}(t,y) \stackrel{\text{def}}{=} u_F^{\mathcal{D}}\big(T-t, \mathrm{Ln}(y)\big), \quad F^{\mathcal{N}}(t,y) \stackrel{\text{def}}{=} u_F^{\mathcal{N}}\big(T-t, \mathrm{Ln}(y)\big), \quad \text{for } y \in [\tfrac{1}{Y}, Y], \tag{33}$$

where $u_F^{\mathcal{D}}$ *is defined by the following problem:*

$$\begin{cases} \dfrac{\partial u_F^{\mathcal{D}}}{\partial \tau} + \mathcal{B}u_F^{\mathcal{D}} = f, \qquad 0 < \tau < T, \forall x \in (-L, +L), \ u_F^{\mathcal{D}}(0,x) = 0, \\ u_F^{\mathcal{D}}(\tau, \pm L) = f \displaystyle\int_0^{\tau} \ e^{-\int_s^{\tau} q_{\pm}(p)dp} e^{-m(\tau-s)} ds. \end{cases} \tag{34}$$

Function $u_F^{\mathcal{N}}$ *is similarly defined but selecting the homogeneous Neumann condition:* $\frac{\partial u_F^{\mathcal{N}}}{\partial x}(\tau, -L) = \frac{\partial u_F^{\mathcal{N}}}{\partial x}(\tau, L) = 0.$

The function $G(t,y)$ is approximated by $G^{\mathcal{D}}(t,y)$ or $G^{\mathcal{N}}(t,y)$ where any degree of accuracy is attained by selecting $Y = e^L$ to be large enough. It is out of scope to theoretically prove the convergence, but numerical results will be presented to confirm that. Other approximations for $G(t,y)$ can be obtained by selecting mixed boundary conditions (Dirichlet condition on one boundary and Neumann on the other), but we only focus here on $G^{\mathcal{D}}(t,y)$ or $G^{\mathcal{N}}(t,y)$. Similar notes apply for $F(t,y)$ which can be approximated by $F^{\mathcal{D}}(t,y)$ or $F^{\mathcal{N}}(t,y)$.

As we will see in the next section, the advantage of selecting homogeneous boundary conditions (where zero-value conditions are imposed), either Dirichlet or Neumann conditions, greatly simplifies the analytic expressions.

6. Time independent coefficients: the eigenfunctions expansion method

Assuming $\boldsymbol{\sigma}$, $\boldsymbol{r}$ and $\boldsymbol{q}$ in (24) to be independent of the time variable, we express $G^{\mathcal{D}}(t,y)$ and $G^{\mathcal{N}}(t,y)$ defined in (32) (respectively, $F^{\mathcal{D}}(t,y)$ and $F^{\mathcal{N}}(t,y)$ in (33)) as eigenfunction expansions based on eigenfunctions of Sturm-Liouville operators, with the corresponding boundary conditions. Considering a finite number of terms in the eigenfunction expansions provides analytic expressions to use in practice to approximate $G(t,y)$ and $F(t,y)$.

The analytic expressions for $G^{\mathcal{N}}(t,y)$ and $F^{\mathcal{N}}(t,y)$ are simpler than those for $G^{\mathcal{D}}(t,y)$ and $F^{\mathcal{D}}(t,y)$, due to the use of a homogeneous boundary conditions versus inhomogeneous ones. We provide expressions for $G^{\mathcal{D}}$ and $F^{\mathcal{D}}$ for the purpose of completeness, especially that the homogeneous Neumann conditions may not applies in the general case, such as when we have a different final payoffs than in (1) for $G^{\mathcal{N}}(t,y)$, or when f depends on the state variable for $F^{\mathcal{N}}(t,y)$.

<u>1/ Eigenfunction expansions for $G^{\mathcal{D}}(t,y)$ and $G^{\mathcal{N}}(t,y)$</u>: Let $A^{\mathcal{D}}$ be the Sturm-Liouville operator $\mathcal{A}$ defined in (23) with the zero-Dirichlet boundary conditions $u(-L) = u(L) = 0$. It is well known (see [**18**]) that A^D is self-adjoint in the Hilbert space $L^2(-L, +L; e^{\alpha x}dx)$ of measurable real-valued functions, equipped with the following inner product:

$$\langle u, v \rangle_{\alpha} = \int_{-L}^{+L} u(x)v(x)e^{-\alpha x}dx, \ \text{ where } \alpha \stackrel{\text{def}}{=} 1 - 2\frac{r-m}{\sigma^2}. \tag{35}$$

where σ, r and q are defined in (24). The operator $A^{\mathcal{D}}$ has a discrete set $\lambda_i^{\mathcal{D}}, i \geq 1$, of eigenvalues that converge to $+\infty$, and the corresponding eigenfunctions $w_i^{\mathcal{D}}, i \geq 1$, form an orthonormal basis so that

$$\langle w_i^{\mathcal{D}}, w_i^{\mathcal{D}} \rangle_\alpha = 1, \qquad \forall i \geq 1, \qquad \text{and} \quad \langle w_j^{\mathcal{D}}, w_i^{\mathcal{D}} \rangle_\alpha = 0, \forall j \neq i.$$

To see why operator $A^{\mathcal{D}}$ is self-adjoint, we use the following integrations by parts:

$$\langle \mathcal{A}u, v \rangle_\alpha = \frac{\sigma^2}{2} e^{-\alpha L} u(L) \frac{\partial v(L)}{\partial x} - \frac{\sigma^2}{2} e^{\alpha L} u(-L) \frac{\partial v(-L)}{\partial x} + \langle u, \mathcal{A}v \rangle_\alpha, \tag{36}$$

for any smooth functions u and v, with v satisfying the zero Dirichlet boundary conditions on $x = -L$ and $x = L$. When u also satisfies the zero Dirichlet boundary conditions we then have $\langle A^{\mathcal{D}}u, v \rangle_\alpha = \langle u, A^{\mathcal{D}}v \rangle_\alpha$.

Likewise, denote by $w_i^{\mathcal{N}}, i \geq 1$, the orthonormal basis of eigenfunctions of the Sturm-Liouville operator $\mathcal{A}$ with the zero-Neumann boundary conditions: $\frac{\partial u(-L)}{\partial x} = \frac{\partial u(L)}{\partial x} = 0$. Let $\lambda_i^{\mathcal{N}}, i \geq 1$, be the set of corresponding eigenvalues. We have the following theorem.

Theorem 1: *Function $G^{\mathcal{D}}(t, y)$ in (32) is such that function $u_c^{\mathcal{D}}(\tau, x)$ defined for $x \in (-L, L)$ is given by the following eigenfunction expansion, where $L > 0$ is a given parameter chosen to be large enough:*

$$\begin{aligned} u_c^{\mathcal{D}}(\tau, x) = &\sum_{i=1}^{+\infty} \langle u_c^{\mathcal{D}}(0, .), w_i^{\mathcal{D}} \rangle_\alpha \, e^{-\lambda_i^{\mathcal{D}} \tau} \, w_i^{\mathcal{D}}(x) \\ &+ \frac{\sigma^2}{2} e^{\alpha L} (K - e^{-L}) e^{-(r+q_-)\tau} \sum_{i=1}^{+\infty} \frac{\partial w_i^{\mathcal{D}}(-L)}{\partial x} \, \frac{1 - e^{-(\lambda_i^{\mathcal{D}} - r - q_-)\tau}}{\lambda_i^{\mathcal{D}} - r - q_-} \, w_i^{\mathcal{D}}(x), \end{aligned} \tag{37}$$

where $u_c^{\mathcal{D}}(0, x) = \mathrm{Max}(0, K - e^x)$. We also have the following for function $G^{\mathcal{N}}(t, y)$ defined in (32):

$$u_c^{\mathcal{N}}(\tau, x) = \sum_{i=1}^{+\infty} \langle u_c^{\mathcal{D}}(0, x), w_i^{\mathcal{N}} \rangle_\alpha e^{-\lambda_i^{\mathcal{N}} \tau} \, w_i^{\mathcal{N}}(x). \tag{38}$$

Proof: Expression (38) can be theoretically justified using the Galerkin approximation as in [**7**]. In fact, it is straightforward to verify that each term of the series in (38) satisfies (31), including the indicated boundary conditions.

For (37), let us note that $u_c^{\mathcal{D}}(\tau, x)$ does not satisfy a zero boundary condition at $x = -L$, however $e^{(r+q_-)\tau} u_c^{\mathcal{D}}(\tau, x)$ satisfies a constant boundary condition at $x = -L$. The derivative with respect to τ, denoted

$$v(\tau, x) \stackrel{\text{def}}{=} \frac{\partial e^{(r+q_-)\tau} u_c^{\mathcal{D}}(\tau, x)}{\partial \tau} = e^{(r+q_-)\tau} \left(\frac{\partial u_c^{\mathcal{D}}(\tau, x)}{\partial \tau} + (r + q_-) u_c^{\mathcal{D}}(\tau, x) \right),$$

then satisfies the zero Dirichlet boundary conditions at both $+L$ and $-L$. From equation (30) the initial value $v_0(x) \stackrel{\text{def}}{=} v(0, x)$ is then given by

$$v_0(.) = -(\mathcal{A} - r - q_-) u_c^{\mathcal{D}}(0, .). \tag{39}$$

Taking the derivative with respect to τ in equation (30) we also get

$$\begin{cases} \dfrac{\partial v}{\partial \tau} + (\mathcal{A} - r - q_-) v = 0, \qquad 0 < \tau < T, \forall x \in (-L, L), \\ v(0, x) = v_0(x) \quad \text{ and } \quad v(\tau, \pm L) = 0, \end{cases}$$

which has homogeneous Dirichlet boundary conditions, and using (39) can then be solved using the following eigenfunction expansion:

$$v(\tau,x)=\sum_{i=1}^{+\infty}\gamma_i e^{-(\lambda_i^{\mathcal{D}}-r-q_-)\tau}w_i^{\mathcal{D}}(x),\ \text{ where }\ \gamma_i=-\langle(\mathcal{A}-r-q_-)u_c^{\mathcal{D}}(0,.),w_i^{\mathcal{D}}\rangle_\alpha.$$

Using formula (36), knowing that $w_i^{\mathcal{D}}$ satisfies the zero Dirichlet boundary conditions $w_i^{\mathcal{D}}(\pm L)=0$, we have

$$\gamma_i=-\frac{\sigma^2}{2}\frac{\partial w_i^{\mathcal{D}}(L)}{\partial x}e^{-\alpha L}u_c^{\mathcal{D}}(0,L)+\frac{\sigma^2}{2}\frac{\partial w_i^{\mathcal{D}}(-L)}{\partial x}e^{\alpha L}u_c^{\mathcal{D}}(0,-L)-\langle u_c^{\mathcal{D}}(0,.),(\mathcal{A}-r-q_-)w_i^{\mathcal{D}}\rangle_\alpha.$$

Since $u_c^{\mathcal{D}}(0,L)=0$ we then have:

$$\gamma_i=\frac{\sigma^2}{2}\frac{\partial w_i^{\mathcal{D}}(-L)}{\partial x}e^{\alpha L}u_c^{\mathcal{D}}(0,-L)-\left(\lambda_i^{\mathcal{D}}-r-q_-\right)\langle u_c^{\mathcal{D}}(0,.),w_i^{\mathcal{D}}\rangle_\alpha.$$

Now, integrating with respect to τ we get:

$$u_c^{\mathcal{D}}(\tau,x)=e^{-(r+q_-)\tau}\left(u_c^{\mathcal{D}}(0,x)+\int_0^\tau v(s,x)ds\right)=u_c^{\mathcal{D}}(0,x)e^{-(r+q_-)\tau}+e^{-(r+q_-)\tau}\sum_{i=1}^{+\infty}\gamma_i\,\frac{1-e^{-(\lambda_i^{\mathcal{D}}-r-q_-)\tau}}{\lambda_i^{\mathcal{D}}-r-q_-}w_i^{\mathcal{D}}(x).$$

On the other hand, note that $u_c^{\mathcal{D}}(0,.)=\sum_{i=1}^{+\infty}\langle u_c^{\mathcal{D}}(0,.),w_i^{\mathcal{D}}\rangle_\alpha w_i^{\mathcal{D}}$, and $u_c^{\mathcal{D}}(0,-L)=K-e^{-L}$ (for $L>0$ large enough), which combined with the equation above gives (37). This completes the proof. **Q.E.D**.

2/ Eigenfunction expansions for $F^{\mathcal{D}}(t,y)$ and $F^{\mathcal{N}}(t,y)$: Let $B^{\mathcal{D}}$ be the Sturm-Liouville operator corresponding to the operator $\mathcal{B}$ defined in (27) with the zero-Dirichlet conditions: $u(-L)=u(L)=0$. Similar to $A^{\mathcal{D}}$, there exists an orthonormal basis $\phi_i^{\mathcal{D}}$, $i\geq 1$, of eigenfunctions of $B^{\mathcal{D}}$ with respect to the Hilbert space $L^2\left(-L,+L;e^{\beta x}dx\right)$ of measured real-valued functions, equipped with the following inner product:

$$\langle u,v\rangle_\beta=\int_{-L}^{+L}u(x)v(x)e^{\beta x}dx\,,\ \text{ where }\beta\stackrel{\text{def}}{=}1+2\frac{r-m}{\sigma^2}. \tag{40}$$

We denote by $\omega_i^{\mathcal{D}}, i\geq 1$, the set of corresponding eigenvalues. Likewise, denote by $\phi_i^{\mathcal{N}}, i\geq 1$, the orthonormal basis of eigenfunction corresponding to the zero-Neumann boundary conditions: $\frac{\partial u(-L)}{\partial x}=\frac{\partial u(L)}{\partial x}=0$. And let $\omega_i^{\mathcal{N}}$, $i\geq 1$, be the set of corresponding eigenvalues. To simplify the notations in the following theorem we introduce the function of three variables below:

$$M(\tau,\delta,\omega)=\frac{1-e^{-(m+\delta)\tau}}{(m+\delta)(\omega-m-\delta)}-\frac{1-e^{-\omega\tau}}{\omega(\omega-m-\delta)}. \tag{41}$$

Theorem 2: *Function $F^{\mathcal{D}}(t,y)$ in (33) is such that function $u_F^{\mathcal{D}}(\tau,x)$ defined for $x\in(-L,L)$ is given by the following eigenfunction expansion, where $L>0$ is a given parameter chosen to be large enough:*

$$\begin{aligned}u_F^{\mathcal{D}}(\tau,x)=&\sum_{i=1}^{+\infty}\langle f,\phi_i^{\mathcal{D}}\rangle_\beta\frac{1-e^{-\omega_i^{\mathcal{D}}\tau}}{\omega_i^{\mathcal{D}}}\phi_i^{\mathcal{D}}(x)\\&+\frac{\sigma^2 f}{2}\sum_{i=1}^{+\infty}\left(e^{-\beta L}\frac{\partial\phi_i^{\mathcal{D}}(-L)}{\partial x}M\left(\tau,q_-,\omega_i^{\mathcal{D}}\right)-e^{\beta L}\frac{\partial\phi_i^{\mathcal{D}}(L)}{\partial x}M\left(\tau,q_+,\omega_i^{\mathcal{D}}\right)\right)\phi_i^{\mathcal{D}}(x)\end{aligned}\tag{42}$$

We also have the following for function $F^{\mathcal{N}}(t,y)$ defined in (33):

$$u_F^{\mathcal{N}}(\tau,x)=\sum_{i=1}^{+\infty}\langle f,\phi_i^{\mathcal{N}}\rangle_\beta\ \frac{1-e^{-\omega_i^{\mathcal{N}}\tau}}{\omega_i^{\mathcal{N}}}\phi_i^{\mathcal{N}}(x). \tag{43}$$

Proof: Expression (43) is easier to prove than (42), so we focus on the latter. Let us introduce the function $v=\frac{\partial u_f^{\mathcal{D}}}{\partial\tau}$. Taking the derivative with respect to τ in equation (34) then gives:

$$\begin{cases}\dfrac{\partial v}{\partial\tau}+\mathcal{B}v=0, \qquad 0<\tau<T, \qquad \forall x\in(-L,L),\\ v(\tau,\pm L)=fe^{-(q_\pm+m)\tau}\ \text{ and }\quad v(0,.)=f-\mathcal{B}\underbrace{u_f^{\mathcal{D}}(0,.)}_{\stackrel{\text{def}}{=}0}=f.\end{cases}$$

We can express v as a sum of two functions $v=v_-+v_+$ where

$$\frac{\partial v_-}{\partial\tau}+\mathcal{B}v_-=0, \qquad \text{with: } v_-(-L,\tau)=fe^{-(q_-+m)\tau} \text{ and } v_-(+L,\tau)=0,$$

and

$$\frac{\partial v_+}{\partial\tau}+\mathcal{B}v_+=0, \qquad \text{with: } v_+(-L,\tau)=0 \text{ and } v_-(+L,\tau)=fe^{-(q_++m)\tau}.$$

Each of the functions v_- and v_+ can be expressed as an eigenfunction expansion following the same argument as for (37). Given that $v_-(0,.)+v_+(0,.)=v(0,.)=f$, we then have:

$$v(\tau,x)=\sum_{i=1}^{+\infty}\langle f,\phi_i^{\mathcal{D}}\rangle_\beta e^{-\omega_i^{\mathcal{D}}\tau}\phi_i^{\mathcal{D}}(x)$$
$$+\frac{\sigma^2 f}{2}\sum_{i=1}^{+\infty}\left(e^{-\beta L}\frac{\partial\phi_i^{\mathcal{D}}(-L)}{\partial x}\ \frac{e^{-(m+q_-)\tau}-e^{-\omega_i^{\mathcal{D}}\tau}}{\omega_i^{\mathcal{D}}-m-q_-}-e^{\beta L}\frac{\partial\phi_i^{\mathcal{D}}(L)}{\partial x}\ \frac{e^{-(m+q_+)\tau}-e^{-\omega_i^{\mathcal{D}}\tau}}{\omega_i^{\mathcal{D}}-m-q_+}\right)\phi_i^{\mathcal{D}}(x)\,.$$

By integrating with respect to τ we have $u_F^{\mathcal{D}}(\tau,x)=\int_0^\tau v(s,x)ds$, since $u_F^{\mathcal{D}}(0,.)=0$, which yields (42) and completes the proof of the theorem. **Q.E.D**.

7. Time dependent coefficients: the recursive formula

This section builds on results from the previous one to propose a recursive formula to determine analytic formulas when σ, r and q, defined in (24), are stepwise constant with respect to τ. In the more general case where σ, r and q are stepwise continuous with respect to τ, we approximate them with stepwise constant functions and use the corresponding analytic formulas. For simplicity we only focus on the recursive formulas for $G^{\mathcal{N}}(\tau,y)$ and $F^{\mathcal{N}}(\tau,y)$ defined in (32) and (33). For the purpose of completeness the formulas for $G^{\mathcal{D}}(\tau,y)$ and $F^{\mathcal{D}}(\tau,y)$, which use more cumbersome notations, will be presented in the appendix.

Case where σ, r and q are stepwise constant with respect to τ: Let $0=\tau_0<\tau_1,\dots,<\tau_{N-1}<\tau_N=T$, be a subdivision of the interval $[0,T]$, for a certain large integer $N>0$. Assume that

$$\sigma(\tau)=\sigma_n, \qquad r(\tau)=r_n, \qquad q(\tau,x)=q_n(x), \quad \forall\tau\in[\tau_n,\tau_{n+1}[\,, \quad n=0,\dots,N-1, \tag{44}$$

where $\sigma_n=\sigma(\tau_n)$, $r_n=r(\tau_n)$ are constants, and $q_n(x)=q(\tau_n,x)$ is a function of x that is independent of τ.

We use similar notations as in the previous section, but with the added indexation with n: we use the same definitions but with σ, r and q, respectively, replaced by σ_n, r_n and $q_n(x)$. For instance, for a given $n < N$, $A_n^{\mathcal{N}}$, is the Sturm-Liouville operator $\mathcal{A}(\tau_n)$ defined in (23) corresponding to the coefficients σ_n, r_n and $q_n(x)$, with the zero-Neumann boundary conditions. We denote by $\lambda_i^{\mathcal{N},n}$, $w_i^{\mathcal{N},n}, i \geq 1$, the corresponding eigenvalues and eigenfunctions. The latter forms an orthonormal basis with respect to the following inner product:

$$\langle u, v \rangle_{\alpha_n} = \int_{-L}^{+L} u(x)v(x)e^{-\alpha_n x}dx\,, \quad \text{where} \quad \alpha_n \stackrel{\text{def}}{=} 1 - 2\frac{r_n - m}{\sigma_n^2}.$$

Theorem 3: *For $n = 0, \dots, N-1$, the function $u_c^{\mathcal{N}}(\tau, x)$, for $\tau \in [\tau_n, \tau_{n+1}]$, is as follows:*

$$u_c^{\mathcal{N}}(\tau, x) = \sum_{i=1}^{+\infty} \xi_i^{\mathcal{N},n} e^{-\lambda_i^{\mathcal{N},n}(\tau - \tau_n)} w_i^{\mathcal{N},n}(x), \tag{45}$$

which determines $G^{\mathcal{N}}(t, y)$ in (32), with $t = T - \tau$ and $y = e^x$. Here $\xi_i^{\mathcal{N},n}$, $i \geq 1$, $n = 0, \dots, N-1$, is recursively defined as follows:

$$\xi_i^{\mathcal{N},0} = \langle \text{Max}(0, K - e^x), w_i^{\mathcal{N},0} \rangle_{\alpha_0} \text{ and } \xi_i^{\mathcal{N},n} = \sum_{j=1}^{+\infty} \langle w_j^{\mathcal{N},n-1}, w_i^{\mathcal{N},n} \rangle_{\alpha_n} \xi_j^{\mathcal{N},n-1}\, e^{-\lambda_j^{\mathcal{N},n-1}(\tau_n - \tau_{n-1})}, \quad i \geq 1\,. \tag{46}$$

Proof: The proof is based on the induction argument. For $n = 0$, formula (45) follows directly from (38). Assume that (45) hold for a given integer $n - 1$, so that

$$u_c^{\mathcal{N}}(\tau_n, x) = \sum_{j=1}^{+\infty} \xi_j^{\mathcal{N},n-1} e^{-\lambda_j^{\mathcal{N},n-1}(\tau_n - \tau_{n-1})} w_j^{\mathcal{N},n-1}(x). \tag{47}$$

We use the fact that function $u_c^{\mathcal{N}}(\tau, x)$ can be determined for $\tau > \tau_n$ by solving equation (31) with the initial condition $h = u_c^{\mathcal{N}}(\tau_n, .)$, so that similar to (38) we have:

$$u_c^{\mathcal{N}}(\tau, x) = \sum_{i=1}^{+\infty} \xi_i^{\mathcal{N},n} e^{-\lambda_i^{n}(\tau - \tau_n)} w_i^{\mathcal{N},n}(x), \quad \text{with} \quad \xi_i^{\mathcal{N},n} = \langle u_c^{\mathcal{N}}(\tau_n, .), w_i^{\mathcal{N},n} \rangle_{\alpha_n}, \qquad \forall i \geq 1.$$

Combining this formula with (47) then gives (45) and (46), which completes the induction argument and the proof of the theorem. **Q.E.D**.

Theorem 4 below provides a recursive formula for $F^{\mathcal{N}}(\tau, x)$. For any given $n < N$, denote by $\omega_i^{\mathcal{N},n}$, $i \geq 1$, the set of eigenvalues of $B_n^{\mathcal{N}}$: the Sturm-Liouville operator $\mathcal{B}(\tau_n)$ defined in (27) corresponding to the coefficients σ_n, r_n and $q_n(x)$, with the zero Neumann boundary conditions at $x = -L$ and $x = L$. $\phi_i^{\mathcal{N},n}$, $i \geq 1$, denote the corresponding eigenfunctions, which form an orthonormal basis with respect to the following inner product:

$$\langle u, v \rangle_{\beta_n} = \int_{-L}^{+L} u(x)v(x)e^{\beta_n x}dx\,, \quad \text{where} \quad \beta_n \stackrel{\text{def}}{=} 1 + 2\frac{r_n - m}{\sigma_n^2}. \tag{48}$$

Theorem 4: *For $n = 0, \dots, N-1$, the function $u_F^{\mathcal{N}}(\tau, x)$, for $\tau \in [\tau_n, \tau_{n+1}]$, is as follows:*

$$u_F^{\mathcal{N}}(\tau, x) = \sum_{i=1}^{+\infty} \left(\vartheta_i^{\mathcal{N},n} e^{-\omega_i^{\mathcal{N},n}(\tau - \tau_n)} + \langle f, \phi_i^{\mathcal{N},n} \rangle_{\beta_n} \frac{1 - e^{-\omega_i^{\mathcal{N},n}(\tau - \tau_n)}}{\omega_i^{\mathcal{N},n}} \right) \phi_i^{\mathcal{N},n}(x), \tag{49}$$

which determines $F^{\mathcal{N}}(t,y)$ from (33), with $t=T-\tau$ and $y=e^x$. Here $\vartheta_i^{\mathcal{N},n}, i\geq 1, n=0,\dots,N-1$, is recursively defined such that $\vartheta_i^{\mathcal{N},0}=0$, and for $n\geq 1$:

$$\vartheta_i^{\mathcal{N},n}=\sum_{j=1}^{+\infty}\langle\phi_j^{\mathcal{N},n-1},\phi_i^{\mathcal{N},n}\rangle_{\beta_n}\left(\vartheta_j^{\mathcal{N},n-1}\,e^{-\omega_j^{\mathcal{N},n-1}(\tau_n-\tau_{n-1})}+\langle f,\phi_j^{\mathcal{N},n-1}\rangle_{\beta_{n-1}}\frac{1-e^{-\omega_j^{\mathcal{N},n-1}(\tau_n-\tau_{n-1})}}{\omega_j^{\mathcal{N},n-1}}\right). \tag{50}$$

Proof: The proof is similarly based on the induction argument. Formula (49), for $n=0$, directly follows from (43). Let us assume that it holds for a given integer $n-1$, so that from (49) we have:

$$u_F^{\mathcal{N}}(\tau_n,x)=\sum_{j=1}^{+\infty}\left(\vartheta_j^{\mathcal{N},n-1}e^{-\omega_j^{\mathcal{N},n-1}(\tau_n-\tau_{n-1})}+\langle f,\phi_j^{\mathcal{N},n-1}\rangle_{\beta_{n-1}}\frac{1-e^{-\omega_j^{\mathcal{N},n-1}(\tau_n-\tau_{n-1})}}{\omega_j^{\mathcal{N},n-1}}\right)\phi_j^{\mathcal{N},n-1}(x), \tag{51}$$

We use the fact that function $u_F^{\mathcal{N}}(\tau,.)$ can be determined for $\tau>\tau_n$ from knowledge of $h=u_F^{\mathcal{N}}(\tau_n,.)$, as $u_F^{\mathcal{N}}(\tau,.)=u_{F,1}(\tau,.)+u_{F,2}(\tau,.)$, where

$$\begin{cases}\dfrac{\partial u_{F,1}}{\partial\tau}+\mathcal{B}(\tau_n)u_{F,1}=f, & \tau>\tau_n,\forall x\in(-L,+L)\\[2ex] u_{F,1}(.,\tau_n)=0\ \text{ and }\ \dfrac{\partial u_{F,1}}{\partial x}(\pm L,\tau)=0, & \end{cases} \tag{52}$$

and

$$\begin{cases}\dfrac{\partial u_{F,2}}{\partial\tau}+\mathcal{B}(\tau_n)u_{F,2}=0, & \tau>\tau_n,\forall x\in(-L,+L),\\[2ex] u_{F,2}(.,\tau_n)=h\ \text{ and }\ \dfrac{\partial u_{F,2}}{\partial x}(\pm L,\tau)=0. & \end{cases} \tag{53}$$

In equations above operator $\mathcal{B}(\tau_n)$ is defined as in (27), corresponding to σ_n, r_n and $q_n(x)$. The solution for $u_{F,1}(\tau,.)$ can be determined similar to (43), providing the terms $\langle f,\phi_i^{\mathcal{N},n}\rangle_{\beta_n}\frac{1-e^{-\omega_i^{\mathcal{N},n}(\tau-\tau_n)}}{\omega_i^{\mathcal{N},n}}$ in equation (49). The solution $u_{F,2}(\tau,.)$ on the other hand can be determined similar to (38) so that

$$u_{F,2}(\tau,x)=\sum_{i=1}^{+\infty}\vartheta_i^{\mathcal{N},n}\,e^{-\omega_i^{\mathcal{N},n}(\tau-\tau_n)}\phi_i^{\mathcal{N},n}(x),\ \text{with}\ \ \vartheta_i^{\mathcal{N},n}=\langle u_F^{\mathcal{N}}(\tau_n,.),\phi_i^{\mathcal{N},n}\rangle_{\beta_n}.$$

This provides the remaining terms in equation (49), and proves the recursive formula (50), given (51). This completes the induction argument and the proof of the theorem. **Q.E.D**.

The general case where σ, r and q are stepwise continuous with respect to τ: Here we approximate σ, r and q with stepwise constant functions, then use recursive formulas (45) and (49). More precisely, assume that σ, r and q are continuous on each interval $[\tau_i,\tau_{i+1}[$, $n=0,\dots,N-1$ in the subdivision of the interval $[0,T]$. Let us define the stepwise constant functions σ^N, r^N and q^N on the interval $[0,T]$ as follows:

$$\sigma^N(\tau)=\left(\frac{1}{\tau_{n+1}-\tau_n}\int_{\tau_n}^{\tau_{n+1}}\sigma^2(s)ds\right)^{1/2},\qquad r^N(\tau)=\frac{1}{\tau_{n+1}-\tau_n}\int_{\tau_n}^{\tau_{n+1}}r(s)ds, \tag{54}$$

$$\text{and}\ \ q^N(\tau,x)=\frac{1}{\tau_{n+1}-\tau_n}\int_{\tau_n}^{\tau_{n+1}}q(s,x)ds,\qquad \forall\tau\in[\tau_n,\tau_{n+1}),\ \text{for}\ n=0,\dots,N-1. \tag{55}$$

Let us denote by $G^{\mathcal{N}}(t,y,\sigma^N,r^N,q^N)$ the analytic expression derived from the recursive formulas (45) and (46), corresponding to the stepwise constant functions σ^N, r^N and q^N. We similarly denote by $F^{\mathcal{N}}(t,y,\sigma^N,r^N,q^N)$ the analytic expression derived from (49) and (50). In [16] it was proved that $G^{\mathcal{N}}(t,y,\sigma^N,r^N,q^N)$ converges to $G^{\mathcal{N}}(t,y)$, as $N \to +\infty$, for a certain norm if the following uniform convergences holds:

$$\underset{\tau,x}{\mathrm{Max}}(|\,\sigma^N(\tau)-\sigma(\tau)|+|\,r^N(\tau)-r(\tau)|+|\,q^N(\tau,x)-q(\tau,x)|) \to 0, \text{as } N \to +\infty, \tag{56}$$

which is the case here since functions σ, r and q are stepwise continuous with respect to τ. Similarly, $F^{\mathcal{N}}(t,y,\sigma^N,r^N,q^N)$ converges to $G^{\mathcal{N}}(t,y)$, as $N \to +\infty$, for a certain norm.

Note that the choice of σ^N, r^N and q^N is not unique. One can for example define $\sigma^N(\tau)$ to be $\sigma(\tau_n)$, and similarly for r^N and q^N, for which the uniform convergence (56) holds. The choice of (54)-(55) however presumably provides a better "convergence" of $G^{\mathcal{N}}(t,y,\sigma^N,r^N,q^N)$ and $F^{\mathcal{N}}(t,y,\sigma^N,r^N,q^N)$, as $N \to +\infty$, (for which we do not provide a formal proof) for the following reasons:

1/ Functions $G^{\mathcal{N}}(t,y,\sigma^N,r^N,q^N)$ and $G^{\mathcal{N}}(t,y)$ are in fact identical if q is independent of x as proved in [16] in a more general case. This in fact always holds when operators $\mathcal{A}(\tau)$ and $\mathcal{A}_s$ (defined by (23)) commute for all $s,\tau \in [\tau_n,\tau_{n+1}]$, and for each n. This commutation property does not exactly hold when q depends on x, but the commutator $\mathcal{A}_\tau\mathcal{A}_s - \mathcal{A}_s\mathcal{A}_\tau$ becomes "smaller" in a sense as $\tau_{n+1}-\tau_n \to 0$.

2/ Electing (54)-(55) corresponds to the first order term in the Magnus expansion as presented in [15], which presumably results in a faster convergence. Using higher order terms could provide an even faster convergence, but this is not within the scope of this article. It may constitute an improvement worth exploring in the future.

8. Numerical illustration and convergence

This section presents a numerical implementation of the analytic formulas established, including the recursive formulas. We assume zero decrements so to assess the convergence by comparing to the exact solutions.

1/ Time-constant coefficients: We consider a GMMB contract issued to a 50 years old policyholder with a time to maturity of 10 years. The policyholder invests in a fund mapped to the S&P/TSX Canadian equity index, and we assume that all parameters as discussed in the previous sections are constant and are as follows:

$$\boldsymbol{r} = 3\%, \qquad \boldsymbol{\sigma} = 15\%, \qquad m = 1\%,\ T = 10 \quad \text{and} \quad K = 1.\ \text{ We also consider } f = 1.$$

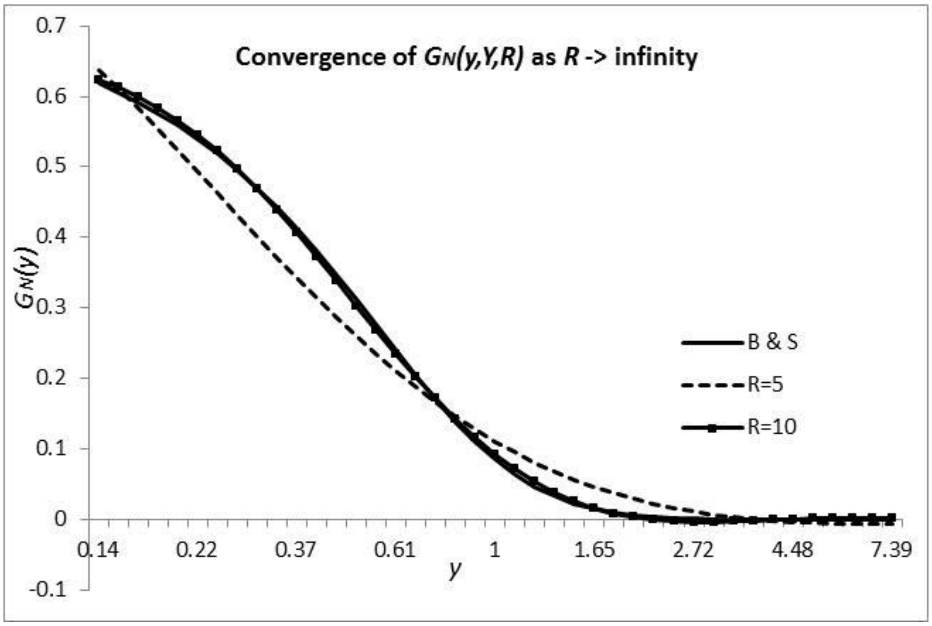
Function $G^{\mathcal{D}}(t,y)$ is defined on the interval $\left(\frac{1}{Y},Y\right)$, for a given parameter $Y > 0$, and we will use the notation $G^{\mathcal{D}}(t,y,Y)$ to indicate the dependence with respect to Y when needed. In addition we only consider in practice the first R terms in the expansion for $G^{\mathcal{D}}(t,y,Y) = u_c^{\mathcal{D}}(T-t,\ln(y),L)$ in (37), with $Y = e^L$, in which case the corresponding analytic expression is denote by $G^{\mathcal{D}}(t,y,Y,R)$ to indicate the dependence with respect to R as well. The analytic expression corresponding to (38), denoted

$$G^{\mathcal{N}}(t,y,Y,R), \quad \text{where } R \text{ is the number of terms in the Series,} \tag{57}$$

is defined in a similar way. The analytic expression $G^{\mathcal{DN}}(t,y,Y,R)$ corresponds to the mixed boundary conditions: Dirichlet type boundary condition for $y = Y$ and Neumann boundary condition for $y = Y^{-1}$.

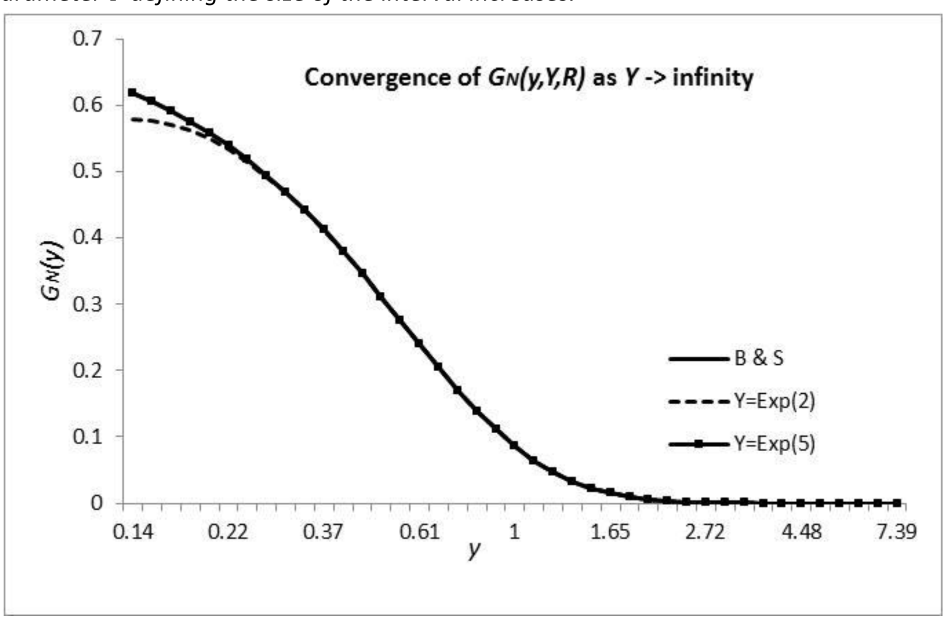
The charts below compare $G^{\mathcal{N}}(t,y,Y,R)$, as a function of y, with $t = 0$, to the exact Black and Scholes solution on the smaller interval $y \in (e^{-2},e^2)$. The two functions should converge as Y and R converge to $+\infty$. In the first chart we assume $Y = e^5$ and show the graphs of $G^{\mathcal{N}}(0, y, Y, R)$ for $R = 5$ and $R = 10$, with the solid line representing the exact solution. Note that an acceptable convergence is achieved at $R = 10$.

Chart 2: *Illustration of the Convergence of $G^{\mathcal{N}}(t, y, Y, R)$ to the exact solution as the number of terms R in the Series increases.*



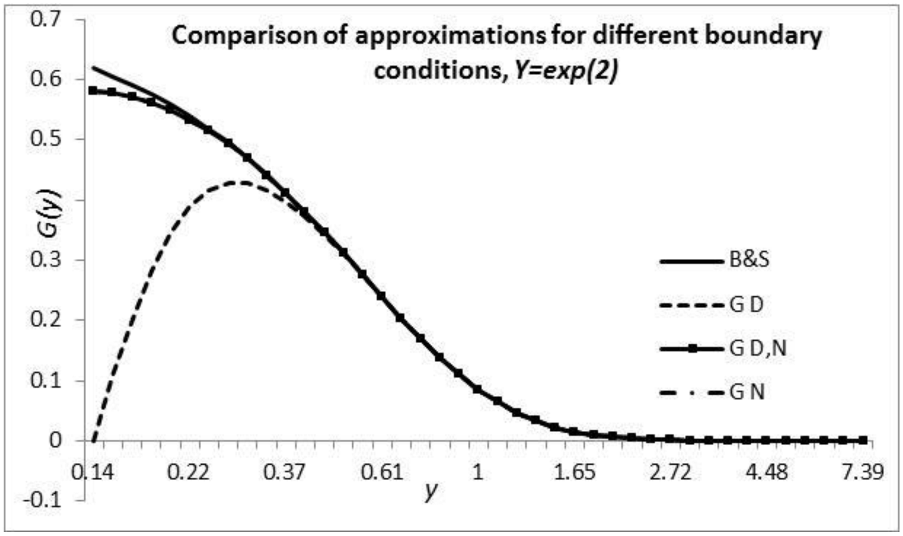
The following chart shows some loss in accuracy for lower values for Y ($Y = e^2$ in this case).

Chart 3: *Illustration of the Convergence of $G^{\mathcal{N}}(t, y, Y, R)$ to the exact solution as the parameter Y defining the size of the interval increases.*



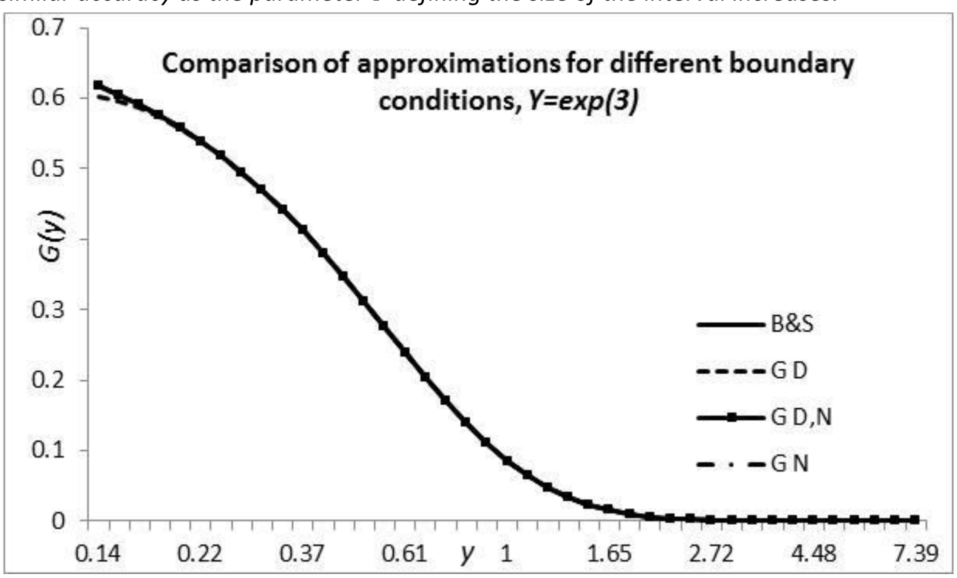
The chart blow compares the approximate solutions corresponding to different boundary conditions with $Y = e^2$. Note that there is a loss in accuracy at lower values of y, especially for $G^{\mathcal{DN}}(t, y, Y, R)$. The latter in fact converges to 0, as $y \to 0$, for any R, because each term in the eigenfunctions expansion satisfy the zero Dirichlet boundary condition for $y = 0$.

Chart 4: *Illustration of the difference in accuracy of the approximate solutions corresponding to different boundary conditions.*



The next chart shows however that the accuracy improves as Y inceases. Note that an acceptable overall accuracy is achieved at only $Y = e^3$.

Chart 5: *Approximate solutions corresponding to different boundary conditions show similar accuracy as the parameter Y defining the size of the interval increases.*



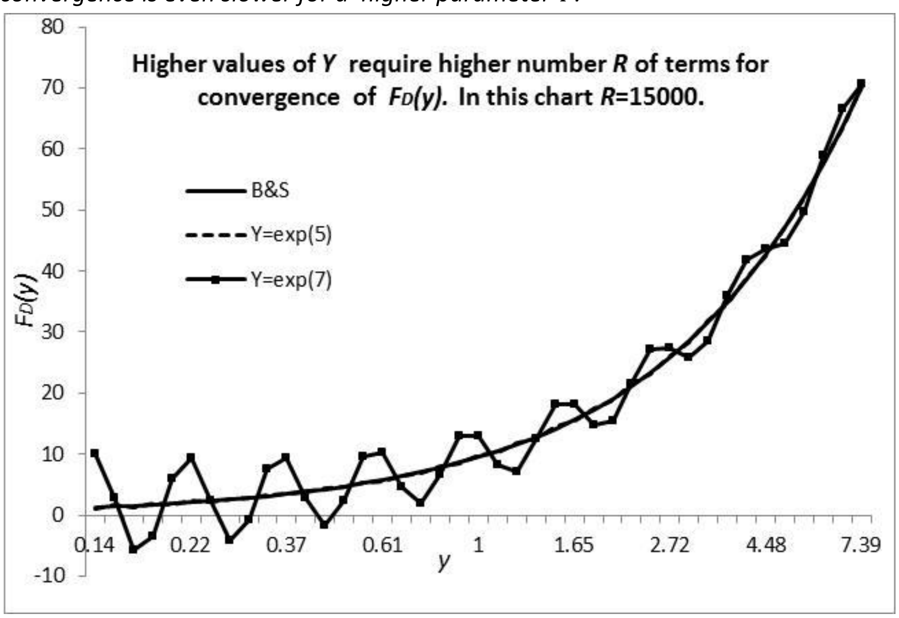
The table below presents approximation errors at different moneyness levels (i.e. different values of $\frac{y}{K}$) and at different times to maturity for $G^{\mathcal{N}}(t, y, Y, R)$. Note the smaller errors for higher $T - t$.

Table 1: Approximation errors for *GN(y)* as function of time-to-maturity *T-t* and number of terms *R*

y	***R=10***			***R=20***		
	T-t=0.5	***T-t=1***	***T-t=10***	***T-t=0.5***	***T-t=1***	***T-t=10***
0.14	0.011	0.011	0.006	0.004	0.003	0.000
0.37	-0.012	-0.013	-0.006	0.004	0.003	0.000
0.82	0.012	0.014	0.004	-0.007	-0.003	0.000
2.72	-0.017	-0.016	-0.004	0.003	0.003	0.000
7.39	0.008	0.007	0.002	0.002	0.001	0.000

We similarly define the analytic expressions $F^{\mathcal{N}}(0,y,Y,R)$ and $F^{\mathcal{D}}(0,y,Y,R)$ for fair market value of the allocated guarantee fee for $f=1$. In the example under consideration function $F^{\mathcal{N}}(t,y,Y,R)$ happens to be identical to the exact solution for only $R=1$. The convergence of function $F^{\mathcal{D}}(0,y,Y,R)$ as R increases is however quite slow, with the convergence being even slower for larger Y as illustrated in the chart below. Note in particular the poor convergence for $Y=e^7$ even though R is close to a large 15,000. This highlights the difference in the rate of convergence of the analytic approximation depending on the boundary condition selected.

Chart 6: *Illustration of the slow convergence of $F^{\mathcal{D}}(0,y,Y,R)$to the exact solution. The convergence is even slower for a higher parameter Y.*



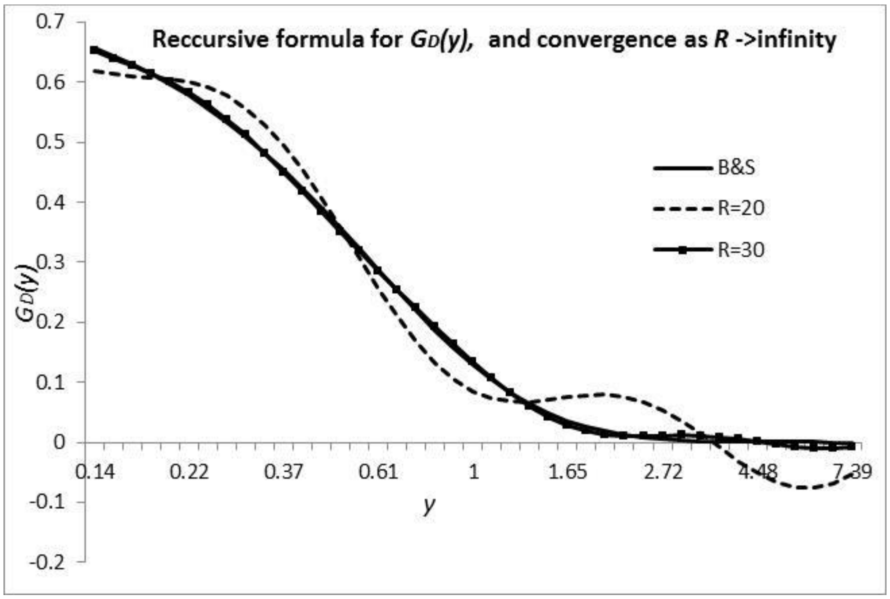
<u>2/ Recursive formulas for the case of time dependent coefficients</u>: We assume the coefficients to be stepwise constant with respect to the time variable, taking two sets of values as follows:

$$\begin{cases} \boldsymbol{r}=3\%, & \boldsymbol{\sigma}=15\% \quad \text{for} \quad t\in[0,5[, \\ \boldsymbol{r}=2\%, & \boldsymbol{\sigma}=20\% \quad \text{for} \quad t\in[5,10], \end{cases}$$

with $m=1\%$, $T=10$ and $K=1$ for $t\in[0,10]$. Note that an exact solution exists based on the Black and Scholes formulas obtained by averaging all the coefficients on the interval $[0,10]$ as proved in [16] in a general case. We implement here the more cumbersome recursive formula for $G^{\mathcal{D}}(t,y,Y)$, as presented in the appendix.

The chart below compares $G^{\mathcal{D}}(0, y, Y, R)$, as a function of y, to the exact Black and Scholes function on the smaller interval (e^{-2}, e^{2}), where $Y = e^{5}$. We show the graphs for $R = 20$ and $R = 30$, with the solid line representing the exact solution.

Chart 7: *Illustration of the recursive formula for $G^{\mathcal{D}}(0, y, Y, R)$ and convergence to the exact solution.*



Note that an acceptable accuracy is achieved for $R = 30$. This is a higher number of terms than in the constant case because of a larger Y considered, and also because there is some loss in accuracy from the recursive formula.

9. Example of analytic expressions when $q(x)$ is stepwise constant, and numerical implementation

This section illustrates how to explicitly calculate eigenfunctions and eigenvalues used in analytic expressions for $G(t, y)$ and $F(t, y)$ in the previous sections in the case where $q(x)$, as defined in (24), is stepwise constant taking two values q_- and q_+ as follows:

$$\begin{cases} q(x) = q_{-,} & x \in (-L, l), \\ q(x) = q_{-,} & x \in (l, L). \end{cases}$$

We assume $K = 1$. The same approach carries over to the case where $q(x)$ takes more than two values. Numerical results for analytic expressions for $G(t, y)$ and $F(t, y)$ as proposed in the previous sections will be illustrated, with all calculations performed in spreadsheet tools. We have assumed that σ, r and q, as defined in (24), are independent of the time variable, otherwise a recursive formula can be used. It is important here to note, from (58) and (61), that when σ and $q(x)$ do not depend on τ the eigenvalues do not need to be numerically recalculated at each iteration in the recursive formula.

Note that assuming $q(x)$ to be piecewise constant with respect to x is not too restrictive in the context of liability modeling for variable annuities. This is because $q(x)$ is often defined so to fit historical lapse and mortality experience, which is generally available in a discrete form. Specific algebraic formats for q may not be required since piecewise constant functions, defined on fine bands of moneyness levels, can provide a good fit for the lapse experience data.

If function $q(x)$ is required to be continuous then a possibility is to impose a certain algebraic formats on q for which eigenfunctions of the corresponding Sturm-Liouville problem can be explicitly expressed in terms of special functions, see [18]. In the case where an algebraic format for q is required one could numerically solve for the eigenvalues and eigenfunctions. Numerical methods are available for that purpose as pointed out in [18].

<u>1/ Analytic expression for $w_i^{\mathcal{N}}$ and $\lambda_i^{\mathcal{N}}$:</u> To simplify the calculations it is useful to note that $\mathcal{A}e^{\frac{\alpha}{2}}w = e^{\frac{\alpha}{2}}\mathcal{A}'w$, were α is given in (35) and

$$\mathcal{A}'w = -\frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} + \left(\frac{\sigma^2}{8}\left(1 - 2\frac{r-m}{\sigma^2}\right)^2 + (r+q)\right)w.$$

We can then look for the eigenfunction $w_i^{\mathcal{N}}$ of $A^{\mathcal{N}}$, corresponding to the eigenvalue $\lambda_i^{\mathcal{N}}$, in the form

$$w_i^{\mathcal{N}} = e^{\frac{\alpha}{2}x}w, \text{ where } -\frac{d^2 w}{dx^2} + \frac{2q}{\sigma^2}w = \nu w, \text{ with } \frac{dw}{dx}(\pm L) + \frac{\alpha}{2}w(\pm L) = 0 \tag{58}$$

and

$$\lambda_i^{\mathcal{N}} = \frac{\sigma^2}{2}\nu + m + \frac{\sigma^2}{2}\left(\frac{1}{2} + \frac{r-m}{\sigma^2}\right)^2. \tag{59}$$

We solve for w and ν in equation (58) separately for $\nu \in \left[\frac{2q_+}{\sigma^2}, +\infty\right]$ and for $\nu \in \left[\frac{2q_-}{\sigma^2}, \frac{2q_+}{\sigma^2}\right]$. We use the fact that q is constant on the interval $(-L, l)$ (respectively, (l, L)) and satisfies the conditions $\frac{\partial w}{\partial x}(\pm L) + \frac{\alpha}{2}w(\pm L) = 0$.

For $\nu \in \left[\frac{2q_+}{\sigma^2}, +\infty\right]$ we have, for some constants E_- and E_+,

$$w(x) = \begin{cases} E_-\left(\sin\big(a_-(x+L)\big) - \frac{2\,a_-}{\alpha}\cos\big(a_-(x+L)\big)\right) & x \in (-L, l) \quad \text{where } a_- = \left(\nu - \frac{2q_-}{\sigma^2}\right)^{\frac{1}{2}} \\ E_+\left(\sin\big(a_+(L-x)\big) + \frac{2\,a_+}{\alpha}\cos\big(a_+(L-x)\big)\right) & x \in (l, L) \quad \text{where } a_+ = \left(\nu - \frac{2q_+}{\sigma^2}\right)^{\frac{1}{2}}. \end{cases}$$

Writing the conditions that w and $\frac{\partial w}{\partial x}$ are continuous at $x = l$ gives a system of two linear equations satisfied by E_- and E_+, which admits a nonzero solution if and only if the following equation is satisfied by ν:

$$\begin{gathered} \frac{\alpha^2 a_+ + 4\,a_-^2 a_+}{\alpha^2}\sin\big(a_-(l+L)\big)\cos\big(a_+(L-l)\big) + \frac{4\,a_+^2 a_- + \alpha^2 a_-}{\alpha^2}\sin\big(a_+(L-l)\big)\cos\big(a_-(l+L)\big) \\ -2\frac{a_+^2 - a_-^2}{\alpha}\sin\big(a_+(L-l)\big)\sin\big(a_-(l+L)\big) = 0. \end{gathered}$$

This equation has an infinite number of solutions $\nu_i,\ i \geq 1$, that converge to $+\infty$. Part of the eigenvalues $\lambda_i^{\mathcal{N}}$ given by (59) can then be determined by numerically solving for the roots of the equation above.

For each $\nu_i,\ i \geq 1$, the coefficients E_- and E_+ are linearly dependent. They need to be scaled so that $w_i^{\mathcal{N}} = e^{\frac{\alpha}{2}x}w$ has a norm of one in the Hilbert space equipped with the scalar product in (35), i.e.

$$\int_{-L}^{L}\left(w_i^{\mathcal{N}}(x)\right)^2 e^{-\alpha x}dx = \int_{-L}^{L} w(x)^2 dx = 1.$$

For $\nu \in \left[\frac{2q_-}{\sigma^2}, \frac{2q_+}{\sigma^2}\right]$ the function w is defined differently for $x \in (l, L)$ by the following formula:

$$w(x) = E'_+\left(\left(\frac{2a'_+}{\alpha}+1\right)e^{a'_+(L-x)} + \left(\frac{2a'_+}{\alpha}-1\right)e^{-a'_+(L-x)}\right),\ \ x \in (l, L)\ \text{ where } a'_+ = \left(\frac{2q_+}{\sigma^2}-\nu\right)^{\frac{1}{2}},$$

where ν satisfies the following equation, which determines the remaining eigenvalues $\lambda_i^{\mathcal{N}}$ thanks to (59):

$$a_-\left(\frac{2a'_+}{\alpha}+1\right)\left(\frac{2a'_+}{\alpha}-1\right)\cos\big(a_-(l+L)\big)\,e^{a'_+(L-l)} - a_-\left(\frac{2a'_+}{\alpha}+1\right)\left(\frac{2a'_+}{\alpha}-1\right)\cos\big(a_-(l+L)\big)\,e^{-a'_+(L-l)}$$

$$-\left(\frac{2a'_+}{\alpha}+1\right)\left(\frac{2\,a_-^2}{\alpha}+a'_+\right)\sin\big(a_-(l+L)\big)\,e^{a'_+(L-l)} - \left(\frac{2a'_+}{\alpha}-1\right)\left(\frac{2\,a_-^2}{\alpha}-a'_+\right)\sin\big(a_-(l+L)\big)\,e^{-a'_+(L-l)} = 0,$$

The coefficients E_- and E'_+ are linearly dependent and need to be scaled so that $w_i^{\mathcal{N}} = e^{\frac{\alpha}{2}x}w$ has a norm of one. To implement formulas (37) and (38) we will also need to determine the following integrals as well, where we assumed $K = 1$, which can be calculated analytically:

$$\int_{-L}^{0}(1-e^x)\,w_i^{\mathcal{N}}(x)\ e^{-\alpha x}dx = \int_{-L}^{0} w(x)\,e^{\frac{-\alpha}{2}x}\,dx - \int_{-L}^{0} w(x)\ e^{\frac{-(\alpha-2)}{2}x}dx.$$

<u>2/ Analytic expression for $\phi_i^{\mathcal{N}}$ and $\omega_i^{\mathcal{N}}$</u>: Similarly, note that $\mathcal{B}e^{-\frac{\beta}{2}x}\phi = e^{-\frac{\beta}{2}x}\mathcal{B}'\phi$, were β is given in (40) and

$$\mathcal{B}'\phi = -\frac{\sigma^2}{2}\frac{\partial^2\phi}{\partial x^2} + \left(\frac{\sigma^2}{8}\left(1+2\frac{r-m}{\sigma^2}\right)^2 + (m+q)\right)\phi.$$

We can then look for the eigenfunction $\phi_i^{\mathcal{N}}$ of $B^{\mathcal{N}}$, corresponding to the eigenvalue $\omega_i^{\mathcal{N}}$, in the form

$$\phi_i^{\mathcal{N}} = e^{-\frac{\beta}{2}x}\phi, \text{and } \ \omega_i^{\mathcal{N}} = \frac{\sigma^2}{2}\kappa + m + \frac{\sigma^2}{2}\left(\frac{1}{2}+\frac{r-m}{\sigma^2}\right)^2, \tag{60}$$

where

$$-\frac{d^2\phi}{dx^2} + \frac{2q}{\sigma^2}\phi = \kappa\phi,\ \text{ and } \frac{d\phi(\pm L)}{dx} - \frac{\beta}{2}\phi(\pm L) = 0. \tag{61}$$

These equations can be solved similarly to the ones above when replacing α by $-\beta$. For $\kappa \in \left[\frac{2q_+}{\sigma^2}, +\infty\right]$ we have, for some constants A_- and A_+,

$$\phi(x) = \begin{cases} A_-\left(\frac{2b_-}{\beta}\cos\big(b_-(x+L)\big) + \sin\big(b_-(x+L)\big)\right),\ \ x \in (-L, l),\ \text{ where } b_- = \left(\kappa - \frac{2q_-}{\sigma^2}\right)^{\frac{1}{2}} \\ A_+\left(-\frac{2b_+}{\beta}\cos\big(b_+(L-x)\big) + \sin\big(b_+(L-x)\big)\right),\ \ x \in (l, L)\ \text{ where } b_+ = \left(\kappa - \frac{2q_+}{\sigma^2}\right)^{\frac{1}{2}}. \end{cases}$$

Writing the conditions for ϕ and $\frac{\partial\phi}{\partial x}$ to be continuous at $x = l$ gives a system of two linear equations satisfied by A_- and A_+, which admits a nonzero solution if and only if the following equation is satisfied by κ:

$$\frac{b_+\beta^2 + 4\, b_+\, b_-^2}{\beta^2} \sin\big(b_-(l+L)\big)\cos\big(\, b_+(L-l)\big) + \frac{b_-\beta^2 + 4\, b_-\, b_+^2}{\beta^2} \cos\big(b_-(l+L)\big)\sin\big(\, b_+(L-l)\big)$$
$$+\, 2\frac{b_+^2 -\, b_-^2}{\beta} \sin\big(b_-(l+L)\big)\sin\big(\, b_+(L-l)\big) = 0.$$

Solving the roots κ_i, $i \geq 0$, of this equation determines a part of the eigenvalues $\omega_i^{\mathcal{N}}$ given by (60), and eigenfunctions $\phi_i^{\mathcal{N}}$, for $i \geq 1$. The coefficients A_- and A_+ are linearly dependent and need to be scaled so that $\phi_i^{\mathcal{N}}(x) = e^{-\frac{\beta}{2}x}\phi(x)$ has a norm of one in the Hilbert space equipped with the scalar product (40), i.e.

$$\int_{-L}^{L} \left(\phi_i^{\mathcal{N}}(x)\right)^2\, e^{\beta x} dx = \int_{-L}^{L} \phi(x)^2 dx = 1.$$

For $\kappa \in \left[\frac{2q_-}{\sigma^2}, \frac{2q_+}{\sigma^2}\right]$ the function ϕ is given by a different formula for $x \in (l, L)$ as follows:

$$\phi(x) = A'_+ \left(\left(\frac{2b'_+}{\beta} - 1\right) e^{b'_+(L-x)} + \left(\frac{2b'_+}{\beta} + 1\right) e^{-b'_+(L-x)} \right),\ \ x \in (l, L)\ \text{ where } b'_+ = \left(\frac{2q_+}{\sigma^2} - \kappa\right)^{\frac{1}{2}},$$

where κ satisfies the following equation, which determines the remaining eigenvalues $\omega_i^{\mathcal{N}}$ given by (60):

$$b_-\left(\frac{2b'_+}{\beta} - 1\right)\left(\frac{2b'_+}{\beta} + 1\right)\cos\big(b_-(l+L)\big)\, e^{b'_+(L-l)} - b_-\left(\frac{2b'_+}{\beta} - 1\right)\left(\frac{2b'_+}{\beta} + 1\right)\cos\big(b_-(l+L)\big)\, e^{-b'_+(L-l)}$$
$$-\left(\frac{2b'_+}{\beta} - 1\right)\left(\frac{2\, b_-^2}{\beta} - b'_+\right)\sin\big(b_-(l+L)\big)\, e^{b'_+(L-l)} - \left(\frac{2b'_+}{\beta} + 1\right)\left(\frac{2\, b_-^2}{\beta} + b'_+\right)\sin\big(b_-(l+L)\big)\, e^{-b'_+(L-l)} = 0.$$

The coefficients A_- and A'_+ are linearly dependent and need to be scaled so that $\phi_i^{\mathcal{N}} = e^{-\frac{\beta}{2}x}\phi$ has a norm of one. To implement formulas (42) and (43) we also analytically calculate following integral:

$$\int_{-L}^{L} f\, \phi_i^{\mathcal{N}}(x)\, e^{\beta x} dx = f \int_{-L}^{L} \phi\,(x)\, e^{\frac{\beta}{2}x} dx.$$

<u>3/ Numerical illustration:</u> Based on the eigenfunctions derived above we numerically illustrate the analytic expression $G^{\mathcal{N}}(0, y, Y, R)$ defined from (57), and $F^{\mathcal{N}}(0, y, Y, R)$ defined from (42), where R is the number of terms considered in the series.

We consider a GMMB contract sold to a 50 years old policyholder with a time to maturity of $T = 10$ years. The fund invests in a 60% constant allocation to the S&P/TSX Canadian equity index, which we assume to have a constant volatility of 20%, and a 40% allocation to the DEX Universe Bond index with an assumed 9% volatility. The two indices are assumed to have a constant correlation of 80%, so from (14) we have $\sigma = 15\%$. We also assume the MER and the interest rates to be constant, respectively equal to $m = 1\%$ and $r = 3\%$.

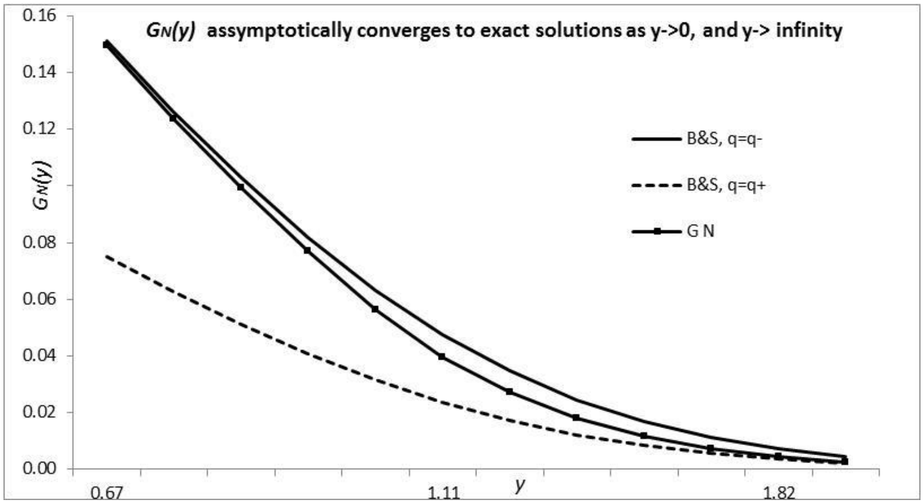
We assume that the mortality rates between the ages of 50 and 60 are zero, and the lapse rate is such that:

$$\begin{cases} \boldsymbol{q} = q_- = 3\%, & \text{for} \quad \dfrac{y}{K} \in [0,1), \\ \boldsymbol{q} = q_+ = 10\%, & \text{for} \quad \dfrac{y}{K} \in [1, +\infty), \end{cases}$$

where y is the account value, and K the guarantee value, at time $t = 0$.

The following chart, for which we assume $K = 1$, compares $G^{\mathcal{N}}(0, y, Y, R)$ as a function of y to the solutions corresponding to the constant decrements of 3% and 10%, respectively, and for which exact solutions exist. An acceptable accuracy for $G^{\mathcal{N}}(0, y, Y, R)$ is attained at $Y = e^5$ and $R = 15$.

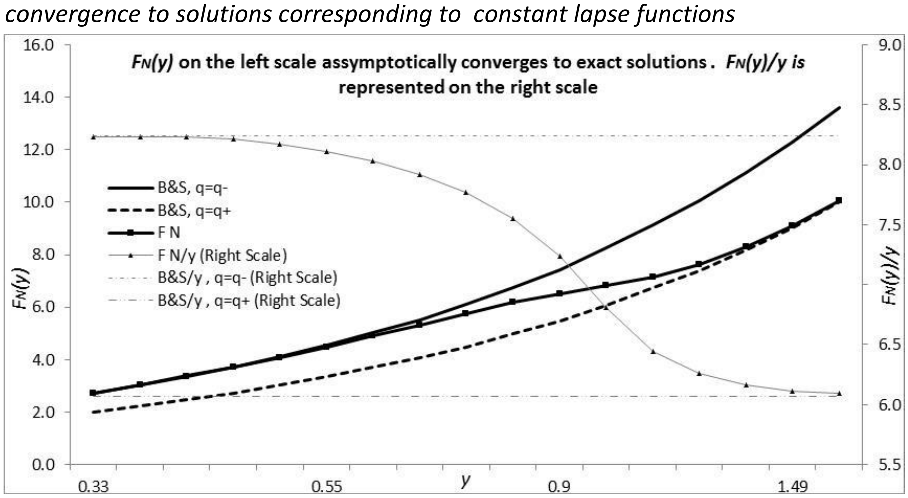
Chart 8: *$G^{\mathcal{N}}(0, y, Y, R)$ with stepwise constant lapse function, and its asymptotic convergence to solutions corresponding to constant lapse functions*



The graph shows that $G^{\mathcal{N}}(0, y, e^5, 15)$ is higher than the function corresponding to the constant lapse rate $q_+ = 10\%$ and converges to it as $y \to +\infty$. It is also lower than the function corresponding to $q_- = 3\%$ and converges to it as $y \to 0$. This in particular reflects the lower cost of guarantee in dollar terms as policyholders surrender their contracts at higher rates.

The next chart compares, on the left scale, $F^{\mathcal{N}}(0, y, Y, R)$, with $f = 1$, as a function of y to the exact solutions corresponding to the constant decrements of 3% and 10%, respectively. An acceptable accuracy for $F^{\mathcal{N}}(0, y, Y, R)$ is attained at $Y = e^5$ and $R = 25$. On the right scale we present a comparison of the previous functions divided by y (the decreasing curve for example represent the graph of the function $\frac{F^{\mathcal{N}}(0, y, e^5, 25)}{y}$).

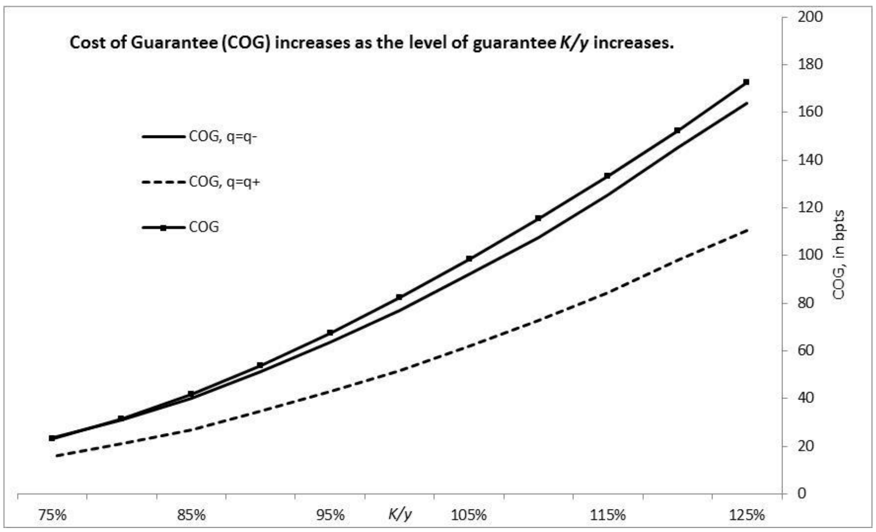
Chart 9: *$F^{\mathcal{N}}(0, y, Y, R)$ with stepwise constant lapse function, and its asymptotic convergence to solutions corresponding to constant lapse functions*



Note that $F^{\mathcal{N}}(t, y, e^5, 25)$ is higher than the value corresponding to the constant lapse rate of $q_+ = 10\%$, and converges to it as $y \to +\infty$. It is also lower than the value corresponding to $q_- = 3\%$, and converges to it as $y \to 0$. This in particular reflects that the fair market value of future allocated fees in dollar terms decreases as the decrements increase.

In the following chart we assume $y = 1$, and present the cost of guarantee in basis points as a function of the level of guarantee $\frac{K}{y} \in [75\%, 125\%]$. By definition, we measured the cost of guarantee by the following ratio: $\mathrm{COG} \stackrel{\text{def}}{=} \frac{G(0,y,K)}{F_{f=1}(0,y,K)}$ as discussed in (10), and is an indication of how expensive the guarantee is.

Chart 10: *Illustration of the increase in the Cost Of Guarantee due to more efficient policyholder behavior as modeled by an increasing dynamic lapse function.*



We see in particular that the COG increases as the level of guarantee increases, and is higher than both measures corresponding to constant lapse rate of $q_- = 3\%$ and $q_+ = 10\%$, respectively, reflecting the higher cost from the policyholder more efficiently using the guarantee by lapsing less when the policy is more in-the-money.

10. Notes on practical implementation, and comparison to SOS Monte Carlo calculations

This section provides notes on the practical implementation of the analytic expressions we developed, and a comparison to the SOS Monte Carlo method where we will see that there is a potential reduction in runtime of the order of $1\mathrm{E}-07$. A more rigorous and detailed comparison is however needed, in particular to provide direct comparisons of runtimes, which will require a more involved numerical study, and is an important future research work.

In practice interest rates are time dependent, and as such the recursive formula in (45) will need to be used. It is then a legitimate question whether the latter will outperform the SOS Monte Carlo calculations. This is the case in our view for the following reasons, and based on the estimation of runtime reduction provided:

- The recursion, in the recursive formula (45), does not add calculation demand compared to the SOS Monte Carlo calculation given that the latter has to be performed at all projected future points in time as well.
- In the recursive formula (45) the intermediate time points $\tau_n, n = 0, \ldots, N-1$, will be monthly at the maximum. This is because in a practical actuarial valuation we assume interest rates to be constant on

monthly time steps. However, using averaging technics as in (54) and (55) we may get acceptable approximations with using an even less refined intermediate time points, such as yearly time steps. See comments provided at the end of section 7.

- For a fair comparison we should compare the total portfolio runtime, and not just the runtime for a single scenario and single policy. The recursive formula (45), contrary to the SOS Monte Carlo method, do not need to be repeated for each scenario and policies (of the same maturity date for example), which greatly saves runtime given that we may have close to a million policies in the portfolio.

Let us now consider a concreate example, and provide an estimate and comparison of the number of calculations required for the recursive formula versus the SOS Monte Carlo method.

Let us consider a portfolio of 1000 policies, all maturing in 10 years. The stochastic valuation is based on 5000 Real World scenarios, and the inner-loop Monte Carlo calculations are based on 1000 Risk Neutral scenarios. The actuarial software used is assumed to project on monthly time-steps. The calculation of the eigenvalues for the analytic formulas is a minor exercise, and is supposed to have been performed prior to the valuation start date.

Number of calculations for the SOS Monte Carlo method: For a single policy, single RN scenario, single RW scenario the number of calculations is $7260 = 120 * 121/2$ given that there are 120 months in the 10 years period. The total number of calculations required is then $7.26E + 13 = 7260 * 1000 * 5000 * 1000 * 2$, given that the calculations have to be performed twice for the determination of the Delta, which is very large number.

Number of calculations using the recursive formula (45): In (46) let us assume that we use matrices $\left(\langle w_j^{\mathcal{N},n-1}, w_i^{\mathcal{N},n}\rangle_{\alpha_n}\right)$ of a $100X100$ size. For the determination of the coefficients $\xi_i^{\mathcal{N},n}$ we are then required to perform a total of $12E + 06 = 120 * 100 * 100$. Based on the numerical example provided is the previous sections it seems enough to only consider 20 terms in the series (45), in which case the total calculations required will be $2.4E + 07 = 12E + 06 * 20$. This is also the total number of calculations required for all policies and scenarios, as the calculations do not need to be repeated for each policy and scenario.

As a conclusion: the number of calculations required to determine the required valuation Delta Greekss is $3.31\text{E} - 07$ smaller when using the recursive formula than when using the full SOS Monte Carlo method. This ratio reduces by 12 if we use yearly time steps instead. It will also dramatically reduce if the number of scenarios, years to maturity and policies increases. It is a very important future research work to perform a detail numerical analysis to compare the two methods more accurately and in particular to compare actual runtimes.

11. Black and Scholes equations for the GMDB, GMWB and GLWB guarantees

For completeness we derive the B&S equations for the fair market value of the cost for the Guaranteed Minimum Death Benefits (GMDB), Guaranteed Minimum Withdrawal Benefits (GMWB) and Guaranteed Life Withdrawal Benefits (GLWB), which are other popular guarantees (also called Riders) offered by variable annuity products.

1/ GMDB: This guarantees the policyholder to receive the higher of the account value and guarantee value at the time of death before maturity T. For a policyholder of age x at policy issue, let $\boldsymbol{q}_d(t)$ be the expected rate of mortality at the attained age $x + t$. We similarly assume that there are no resets, rollups, ratchets and other complex features. We denote by $V(s)$ the value of the account at time $s \geq t$, and by $S(s)$ the value of the underlying fund (i.e. the account value ignoring decrements) such that $V(t) = S(t)$, which are related by (4). As for (6), the expected death claim at time t payable on the interval $(s, s + dt)$, for $t \leq s \leq T$, is given by:

$$Max\left(0, Q(t,s)K - V(s)\right)q_d(s)ds = Q(t,s)Max\left(0, K - S(s)\right)\boldsymbol{q}_d(s)ds,$$

where K is the guarantee value at time t. Similar to (8), the expected total death claim is then given by

$$\int_t^T e^{-\int_t^s \boldsymbol{q}\left(p,\frac{S(p)}{K}\right)dp} Max\left(0, K - S(s)\right)\boldsymbol{q}_d(s)ds.$$

The Fair Market Value at time $t \leq T$ of the expected total death claims between t and T is then given by the following risk neutral valuation expression, similar to (19):

$$G_d(t,y) = \underset{S(t))=y}{\mathbb{E}_{\mathbb{Q}}}\left(\int_t^T e^{-\int_t^s \boldsymbol{r}(p)dp}\ e^{-\int_t^s \boldsymbol{q}\left(p,\frac{S(p)}{K}\right)dp} Max\left(0, K - S(s)\right)\boldsymbol{q}_d(s)ds\right),$$

where y is the value of the account at time t. Using the Feynman-Kac formulas as for (20) we get:

$$\begin{cases} \dfrac{\partial G_d}{\partial t} + (\boldsymbol{r} - m)y\dfrac{\partial G_d}{\partial y} + \dfrac{1}{2}\sigma^2 y^2 \dfrac{\partial^2 G_d}{\partial y^2} - \boldsymbol{r}G_d - \boldsymbol{q}\left(t, \dfrac{y}{K}\right)G_d + Max\left(0, K - y\right)\boldsymbol{q}_d(t) = 0, \qquad 0 < t < T, \forall y > 0, \\ \\ G_d(y,T) = 0,\ \text{as a final condition,} \end{cases}$$

which can be solved using similar techniques as in (42) and (43).

<u>2/ GMWB and GLWB</u>: The GMWB guarantees the policyholder to receive a minimum total dollar amount of withdrawals $K = T\,\theta$, where θ is the assumed continuous rate of yearly withdrawals. In the case where the account value depletes at time t before maturity T, the policyholder is guaranteed to continue to receive payment at the rate of θ, for a total remaining guarantee value of $K(t) = (T - t)\theta$. At maturity T the policyholder receives the remaining fund value, if not already depleted.

If we assume the rate of withdrawals θ to proportionately reduce with decrements then we also have $\theta(s) = Q(t,s)\,\theta(t)$. Using (18) we then have the following stochastic differential equation under a risk neutral measure $\mathbb{Q}$: $dS = \left((\boldsymbol{r} - m)S - \theta\right)dt + S\boldsymbol{\sigma}\,dw$ (we assumed proportional withdrawals from all the account's investment components). We assume the decrement rate $\boldsymbol{q}$ to be dynamic and function of the moneyless level, which we define at time $s \geq t$ by:

$$\frac{V(s)}{(T-s)\,\theta(s)} = \frac{Q(t,s)S(s)}{(T-s)Q(t,s)\,\theta(t)} = \frac{S(s)}{(T-s)\,\theta}, \quad \text{with} \quad \theta = \theta(t).$$

The definition of moneyness above is not unique and could have been based on a different "perceived" remaining guarantee value measure at time t (instead of $(T-s)\theta$). The Fair Market Value at time $t \leq T$ of the expected total withdrawal claims between t and T is then given by the following risk neutral valuation expression, were $y = V(t) = S(t)$ is the value of the account at time t:

$$G_w(t,y) = \underset{S(t)=y}{\mathbb{E}_{\mathbb{Q}}}\left(e^{-\int_t^{\tilde{t}}\left(\boldsymbol{q}\left(p,\frac{S(p)}{(T-p)\,\theta}\right)+\boldsymbol{r}(p)\right)dp}\,\theta\psi(\tilde{t})\right),$$

where $\theta\psi(t')$ is the discounted value of withdrawal claims in case the account depletes for the first time at t':

$$\psi(t') = \theta\int_{t'}^T e^{-\int_{t'}^s \boldsymbol{r}(p)dp} e^{-\int_{t'}^s \boldsymbol{q}_d(p)dp} ds,$$

and $\tilde{t}(t,y) = \inf\{s; s \geq t \text{ and } S(s) \leq 0\}$ is a stopping time for the stochastic variable $s \to S(s)$, and $\boldsymbol{q}_d(s)$ the mortality rate at time s. From this risk neutral expression, and using the Feynman-Kac formulas, we derive the following partial differential equation, with boundary condition at $y = 0$:

$$\begin{cases}\dfrac{\partial G_w}{\partial t}+\left((\boldsymbol{r}-m)y-\theta\right)\dfrac{\partial G_w}{\partial y}+\dfrac{1}{2}\boldsymbol{\sigma}^2y^2\dfrac{\partial^2 G_w}{\partial y^2}-\boldsymbol{r}G_w-\boldsymbol{q}\left(t,\dfrac{y}{(T-t)\theta}\right)G_w=0, & 0<t<T,\forall y>0,\\ G_w(0,t)=\theta\displaystyle\int_t^T e^{-\int_t^s \boldsymbol{r}(p)dp}e^{-\int_t^s \boldsymbol{q}_d(p)dp}ds\,, & \text{as a boundary condition,}\\ G_w(y,T)=0, & \text{as a final condition.}\end{cases}$$

The equation for the fair market value of the GLWB guarantee cost is similar, with $T=100$ (or a higher age if the mortality table used in the liability model extends beyond the age 100).

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Appendix - Other recursive formulas

This appendix completes section 7 by presenting recursive formulas for $G^{\mathcal{D}}(t,y)$ and $F^{\mathcal{D}}(t,y)$, which require more cumbersome notations. These formulas may be useful when zero boundary conditions do not apply, which maybe the case for example when considering a different payoff than in (1), or when the fee f in (2) dependent y. Functions $G^{\mathcal{D}}(t,y)$ and $F^{\mathcal{D}}(t,y)$ are defined on a certain interval $\left[\frac{1}{Y},Y\right]$, for some $Y=e^{L}$.

As in section 7, functions $\sigma(\tau), r(\tau)$ and $q(\tau,x)$ defined in (24) are stepwise constant with respect to $\tau\in[0,T]$, corresponding to a subdivision of $[0\ T]$ as in (44). For each $n=0,\dots,N-1$, we denote by $\lambda_i^{\mathcal{D},n}$, $w_i^{\mathcal{D},n}$, $i\geq 1$, the eigenvalues and eigenfunctions of the Sturm-Liouville operator $\mathcal{A}(\tau_n)$ defined as in (23) corresponding to the coefficients σ_n, r_n and $q_n(x)$, with the zero-Dirichlet boundary conditions. We also denote

$$q_{n,-}=\lim_{x\to-\infty} q(\tau_n,x) \text{ and } q_{n,+}=\lim_{x\to+\infty} q(\tau_n,x), \quad \text{for } n=0,\dots,N-1.$$

We have the following theorem:

Theorem 5: *As for* (45)*, for each* $n=0,\dots,N-1$*, the function* $u_c^{\mathcal{D}}(\tau,x)$*, for* $\tau\in[\tau_n,\tau_{n+1}]$*, is as follows:*

$$\begin{aligned} u_c^{\mathcal{D}}(\tau,x) = &\sum_{i=1}^{+\infty}\xi_i^{\mathcal{D},n} e^{-\lambda_i^{\mathcal{D},n}(\tau-\tau_n)} w_i^{\mathcal{D},n}(x) \\ &+\frac{\sigma_n^2}{2} e^{\alpha_n L} K' e^{-\int_0^{\tau}(r(s)+q_-(s))ds} \sum_{i=1}^{+\infty} \frac{\partial w_i^{\mathcal{D},n}(-L)}{\partial x}\, \frac{1-e^{-\left(\lambda_i^{\mathcal{D},n}-r_n-q_{n,-}\right)(\tau-\tau_n)}}{\lambda_i^{\mathcal{D},n}-r_n-q_{n,-}} w_i^{\mathcal{D},n}(x), \end{aligned} \tag{62}$$

with $K'=(K-e^{-L})$*, which determines* $G^{\mathcal{D}}(t,y)$ *by (32), with* $t=T-\tau$ *and* $y=e^{x}$*. Here* $\xi_i^{\mathcal{D},n}, i\geq 1, n=0,\dots,N-1$*, is recursively defined by* $\xi_i^{\mathcal{D},0}=\langle \mathrm{Max}(0,K-e^{x}), w_i^{\mathcal{D},0}\rangle_{\alpha_0}$ *and*

$$\begin{aligned} \xi_i^{\mathcal{D},n} = \sum_{j=1}^{+\infty} \langle w_j^{\mathcal{D},n-1}, w_i^{\mathcal{D},n}\rangle_{\alpha_n} \Bigg(&\xi_j^{\mathcal{D},n-1} e^{-\lambda_j^{\mathcal{D},n-1}(\tau_n-\tau_{n-1})} \\ &+\frac{\sigma_{n-1}^2}{2} e^{\alpha_{n-1}L} K' e^{-\int_0^{\tau_n}(r(s)+q_-(s))ds}\, \frac{\partial w_j^{\mathcal{D},n-1}(-L)}{\partial x} \frac{1-e^{-\left(\lambda_j^{\mathcal{D},n}-r_{n-1}-q_{n-1,-}\right)(\tau_n-\tau_{n-1})}}{\lambda_j^{\mathcal{D},n-1}-r_{n-1}-q_{n-1,-}} \Bigg), \end{aligned} \tag{63}$$

Proof: The proof is based on the induction argument. The result for $n=0$ is a direct consequence of (37). Assume (62) to hold for $n-1$, so that

$$\begin{aligned} u_c^{\mathcal{D}}(\tau_n,x) = &\sum_{j=1}^{+\infty}\xi_j^{\mathcal{D},n-1} e^{-\lambda_j^{\mathcal{D},n-1}(\tau_n-\tau_{n-1})} w_j^{\mathcal{D},n-1}(x) \\ +&\frac{\sigma_{n-1}^2}{2} e^{\alpha_{n-1}L} K' e^{-\int_0^{\tau_n}(r(s)+q_-(s))ds} \sum_{j=1}^{+\infty} \frac{\partial w_j^{\mathcal{D},n-1}(-L)}{\partial x}\, \frac{1-e^{-\left(\lambda_j^{\mathcal{D},n-1}-r_{n-1}-q_{n-1,-}\right)(\tau_n-\tau_{n-1})}}{\lambda_j^{\mathcal{D},n-1}-r_{n-1}-q_{n-1,-}} w_j^{\mathcal{D},n-1}(x). \end{aligned} \tag{64}$$

Now we use the fact that similar to (37) function $u_c^{\mathcal{D}}(\tau,x)$ can be determined for $\tau > \tau_n$ based on knowledge of $u_c^{\mathcal{D}}(\tau_n,.)$ as follows:

$$\begin{aligned} u_c^{\mathcal{D}}(\tau,.) &= \sum_{i=1}^{+\infty} \xi_i^{\mathcal{D},n}\, e^{-\lambda_i^{\mathcal{D},n}(\tau-\tau_n)} w_i^{\mathcal{D},n}(x) \\ &\quad + \frac{\sigma_n^2}{2} e^{\alpha_n L} u_c^{\mathcal{D}}(\tau_n,-L) e^{-(r_n+q_{n,-})(\tau-\tau_n)} \sum_{i=1}^{+\infty} \frac{\partial w_i^{\mathcal{D},n}(-L)}{\partial x} \frac{1-e^{-\left(\lambda_i^{\mathcal{D},n}-r_n-q_{n,-}\right)(\tau-\tau_n)}}{\lambda_i^{\mathcal{D},n}-r_n-q_{n,-}} w_i^{\mathcal{D},n}(x)\,. \end{aligned} \tag{65}$$

with $\xi_i^{\mathcal{D},n} = \langle u_c^{\mathcal{D}}(\tau_n,.), w_i^{\mathcal{D},n}\rangle_{\alpha_n}, \forall i \geq 1$. Note that from (30) we have

$$u_c^{\mathcal{D}}(\tau_n,-L) e^{-(r_n+q_{n,-})(\tau-\tau_n)} = K'\, e^{-\int_0^{\tau}(r(s)+q_-(s))ds}.$$

Then combing (65), with formula (64), gives (62) and (63), which completes the induction argument and the proof of the theorem. **Q.E.D**.

For the theorem below we similarly, for each $n = 0,\dots,N-1$, denote by $\omega_i^{\mathcal{D},n}$, $\phi_i^{\mathcal{D},n}, i \geq 1$, the eigenvalues and eigenfunctions of the Sturm-Liouville operator $\mathcal{B}(\tau_n)$ defined as in (27) corresponding to the coefficients σ_n, r_n and $q_n(x)$, with the zero-Dirichlet boundary conditions. For ease of the presentation let us introduce the following notations using function $M(.)$ as defined in (41):

$$X_{\pm,i}^n(\tau,\tilde{\tau},\delta,\omega) = \frac{\sigma_n^2}{2}\, f\, e^{\pm\beta_n L} \frac{\partial \phi_i^{\mathcal{D},n}(\pm L)}{\partial x}\left(M(\tau,\delta,\omega) + \frac{e^{-(m+\delta)\tau} - e^{-\omega\tau}}{\omega - m - \delta} \int_0^{\tilde{\tau}} e^{-\int_s^{\tilde{\tau}} q_{\pm}(p)dp} e^{-m(\tilde{\tau}-s)} ds\right).$$

Theorem 6: *For each $n = 0,\dots,N-1$, the function $u_F^{\mathcal{D}}(\tau,x)$, for $\tau \in [\tau_n,\tau_{n+1}]$, is as follows:*

$$\begin{aligned} u_F^{\mathcal{D}}(\tau,x) = \sum_{i=1}^{+\infty} \Bigg(&\vartheta_i^{\mathcal{D},n} e^{-\omega_i^{\mathcal{D},n}(\tau-\tau_n)} + \langle f, \phi_i^{\mathcal{D},n}\rangle_{\beta_n} \frac{1-e^{-\omega_i^{\mathcal{D},n}(\tau-\tau_n)}}{\omega_i^{\mathcal{D},n}} + X_{-,i}^n\left(\tau-\tau_n,\tau_n,q_{n,-},\omega_i^{\mathcal{D},n}\right) \\ &- X_{+,i}^n\left(\tau-\tau_n,\tau_n,q_{n,+},\omega_i^{\mathcal{D},n}\right)\Bigg) \phi_i^{\mathcal{D},n}(x), \end{aligned} \tag{66}$$

which determines $F^{\mathcal{D}}(t,y)$ from (33), with $t = T-\tau$ and $y = e^x$. Here $\vartheta_i^{D,0} = 0,\ i \geq 1,$ and $\vartheta_i^{D,n}$, $n = 1,\dots,N-1$, is recursively defined as follows:

$$\begin{aligned} \vartheta_i^{D,n} = \sum_{j=1}^{+\infty} \langle \phi_j^{\mathcal{D},n-1}, \phi_i^{\mathcal{D},n}\rangle_{\beta_n} \Bigg(&\vartheta_j^{D,n-1} e^{-\omega_j^{\mathcal{D},n-1}(\tau_n-\tau_{n-1})} + \langle f, \phi_j^{\mathcal{D},n-1}\rangle_{\beta_{n-1}} \frac{1-e^{-\omega_j^{\mathcal{D},n-1}(\tau_n-\tau_{n-1})}}{\omega_j^{\mathcal{D},n-1}} \\ &+ X_{-,j}^{n-1}\left(\tau_n-\tau_{n-1},\tau_{n-1},q_{n-1,-},\omega_j^{\mathcal{D},n-1}\right) - X_{+,j}^{n-1}\left(\tau_n-\tau_{n-1},\tau_{n-1},q_{n-1,+},\omega_j^{\mathcal{D},n-1}\right)\Bigg). \end{aligned} \tag{67}$$

Proof: The proof is based on an induction argument. The result for $n = 0$ is a direct consequence of (42). Assume (66) to hold for $n-1$, so that

$$u_F^{\mathcal{D}}(\tau_n,x)=\sum_{j=1}^{+\infty}\left(\vartheta_j^{\mathcal{D},n-1}e^{-\omega_j^{\mathcal{D},n-1}(\tau_n-\tau_{n-1})}+\langle f,\phi_j^{\mathcal{D},n-1}\rangle_{\beta_{n-1}}\frac{1-e^{-\omega_j^{\mathcal{D},n-1}(\tau_n-\tau_{n-1})}}{\omega_j^{\mathcal{D},n-1}}\right.$$
$$\left.+\ X_{-,j}^{n-1}\left(\tau_n-\tau_{n-1},\tau_{n-1},q_{n-1,-},\omega_j^{\mathcal{D},n-1}\right)-X_{+,j}^{n-1}\left(\tau_n-\tau_{n-1},\tau_{n-1},q_{n-1,+},\omega_j^{\mathcal{D},n-1}\right)\right)\phi_j^{\mathcal{D},n-1}(x).$$

Before we proceed let us note that the solution $u_F^{\mathcal{D}}(\tau,.)$ to (34) can be determined for $\tau_{n+1}<\tau\leq\tau_{n+1}$ from knowledge of $h(.)=u_F^{\mathcal{D}}(\tau_n,.)$, as $u_F^{\mathcal{D}}(\tau,x)\ =u_{F,1}(\tau,x)+u_{F,2}(\tau,x)$, where

$$\begin{cases}\dfrac{\partial u_{F,1}}{\partial\tau}+\mathcal{B}(\tau_n)u_{F,1}=f, \qquad \tau_n<\tau\leq\tau_{n+1}, \qquad \forall x\in(-L,+L),\ u_{F,1}(.,\tau_n)=0,\\ u_{F,1}(\pm L,\tau)=f\displaystyle\int_{\tau_n}^{\tau}\ e^{-\int_s^\tau q_{n,\pm}(p)dp}e^{-m(\tau-s)}ds,\end{cases}\tag{68}$$

and

$$\begin{cases}\dfrac{\partial u_{F,2}}{\partial\tau}+\mathcal{B}(\tau_n)u_{F,2}=0, \qquad \tau_n<\tau\leq\tau_{n+1}, \qquad \forall x\in(-L,+L),\ u_{F,2}(\tau_n,.)=h(.),\\ u_{F,2}(\pm L,\tau)=h(\pm L)e^{-\int_{\tau_n}^{\tau}q_{n,\pm}(p)dp}e^{-m(\tau-\tau_n)}.\end{cases}\tag{69}$$

Here operator $\mathcal{B}(\tau_n)$ is defined as in (27), corresponding to the coefficients σ_n,r_n and $q_n(x)$.

To show that $u_F^{\mathcal{D}}(\tau,x)=u_{F,1}(\tau,x)+u_{F,2}(\tau,x)$ satisfy the boundary condition in (34), note first that from (34)

$$h(\pm L)=u_F^{\mathcal{D}}(\tau_n,.)=f\int_0^{\tau_n}\ e^{-\int_s^{\tau_n}q_\pm(p)dp}e^{-m(\tau_n-s)}ds.$$

Then straightforward calculations show the following, which means the boundary condition (34) is satisfied:

$$u_F(\pm L,\tau)=u_{F,1}(\pm L,\tau)+h(\pm L)=f\int_0^{\tau}\ e^{-\int_s^\tau q_\pm(p)dp}e^{-m(\tau-s)}ds\tag{70}$$

Going back to the induction argument, note that solution $u_{F,1}(\tau,x)$ can be solved as in (42) as:

$$u_{F,1}(\tau,x)=\sum_{i=1}^{+\infty}\langle f,\phi_i^{\mathcal{D},n}\rangle_{\beta_n}\frac{1-e^{-\omega_i^{\mathcal{D},n}(\tau-\tau_n)}}{\omega_i^{\mathcal{D},n}}\phi_i^{\mathcal{D},n}(x)+$$
$$\frac{\sigma_n^2}{2}f\sum_{i=1}^{+\infty}\left(e^{-\beta_nL}\frac{\partial\phi_i^{\mathcal{D},n}(-L)}{\partial x}\ M\left(\tau-\tau_n,q_{n,-},\omega_i^{\mathcal{D},n}\right)-e^{\beta_nL}\frac{\partial\phi_i^{\mathcal{D},n}(L)}{\partial x}M\left(\tau-\tau_n,q_{n,+},\omega_i^{\mathcal{D},n}\right)\right)\phi_i^{\mathcal{D},n}(x).\tag{71}$$

Solution $u_{F,2}(\tau,x)$ on the other hand can be solved similar to (37) as:

$$\begin{aligned}u_{F,2}(\tau,x)=&\sum_{i=1}^{+\infty}\vartheta_i^{\mathcal{D},n}e^{-\omega_i^{\mathcal{D},n}(\tau-\tau_n)}\,\phi_i^{\mathcal{D},n}(x)\\&+\ \frac{\sigma_n^2}{2}u_F^{\mathcal{D}}(\tau_n,-L)e^{-\beta_nL}\sum_{i=1}^{+\infty}\frac{\partial\phi_i^{\mathcal{D},n}(-L)}{\partial x}\ \frac{e^{-(m+q_-)(\tau-\tau_n)}-e^{-\omega_i^{\mathcal{D},n}(\tau-\tau_n)}}{\omega_i^{\mathcal{D},n}-m-q_-}\ \phi_i^{\mathcal{D},n}(x)\\&-\frac{\sigma_n^2}{2}u_F^{\mathcal{D}}(\tau_n,L)e^{\beta_nL}\sum_{i=1}^{+\infty}\frac{\partial\phi_i^{\mathcal{D},n}(L)}{\partial x}\ \frac{e^{-(m+q_+)(\tau-\tau_n)}-e^{-\omega_i^{\mathcal{D},n}(\tau-\tau_n)}}{\omega_i^{\mathcal{D},n}-m-q_+}\phi_i^{\mathcal{D},n}\,(x),\end{aligned}\tag{72}$$

with $\vartheta_i^{\mathcal{D},n}=\langle u_F^{\mathcal{D}}(\tau_n,.),\phi_i^{\mathcal{D},n}\rangle$. In the above formula the terms $u_F^{\mathcal{D}}(\tau_n,\pm L)$ are given from the boundary condition (70). Combining this with (71), (72) and the fact that $\vartheta_i^{\mathcal{D},n}=\langle u_F^{\mathcal{D}}(\tau_n,.),\phi_i^{\mathcal{D},n}\rangle$, proves (66) and the recursive formula (67), which completes the induction argument and the proof of the theorem. **Q.E.D**.