Market Volatility Risk in an Era of Extreme Events

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Glossary

A191RL1Q225SBEA: Real Gross Domestic Product, Percent Change from Preceding Period. The name of the data series from the original data source is kept to facilitate data validation and modeling.

ARMA: autoregressive moving average model, a time series model (ARMA) captures autocorrelation of the time variant feature.

AUM_BSB: Asset under management: Balanced (Stock and Bonds)

AUM_Con: Asset under management: Convertible Arbitrage

AUM_Dis: Asset under management: Distressed Securities

AUM_ED: Asset under management: Event Driven

AUM_ELB: Asset under management: Equity Long Bias

AUM_ELO: Asset under management: Equity Long Only

AUM_ELS: Asset under management: Equity Long/Short

AUM_EM: Asset under management: Emerging Markets

AUM_EMA: Asset under management: Emerging Markets - Asia

AUM_EMEE: Asset under management: Emerging Markets – Eastern Europe

AUM_EMG: Asset under management: Emerging Markets – Global

AUM_EMLA: Asset under management: Emerging Markets – Latin America

AUM_EMN: Asset under management: Equity Market Neutral

AUM_FI: Asset under management: Fixed Income

AUM_fof: Asset under management: Fund of Funds

AUM_HF: Asset under management: Hedge Funds, excluding fund of funds assets

AUM_MA: Asset under management: Merger Arbitrage

AUM_Mac: Asset under management: Macro

AUM_MS: Asset under management: Multi-Strategy

AUM_OS: Asset under management: Options Strategies

AUM_Other: Asset under management: Other, including funds categorized as Algorithmic, Closed-end funds, Dividend Capture, Equity Dedicated Short, Equity Short-Bias, Mutual Funds/ETFs, No Category, PIPEs (Regulation D), Replication, and Tail Risk.

AUM_SS: Asset under management: Sector Specific, including sector funds categorized as Energy, Environment, ESG, Farming, Financial, Health Care/Biotech, Metals/Mining, Miscellaneous, Natural Resources, Real Estate, and Technology.

CART: Classification and Regression Tree models are a basic form of tree-based models. CART models build trees to split the data based on explanatory variables. At each split, a variable is used to separate the data into two subgroups. The variable is chosen to provide the best split that improves the purity of the data in the subgroups.

Copula: A copula is used to formulate a multivariate distribution via a simple transformation being made of each marginal variable in such a way that each transformed marginal variable has a uniform distribution. Its theoretical foundation is Sklar's theorem of 1959, which says that every multivariate CDF can be written as a function of the marginal distribution functions.

covid_case_us: U.S. daily COVID cases

covid_death_us: U.S. daily COVID deaths

CPIAUCSL: Consumer Price Index for All Urban Consumers

DEF: Federal Funds Effective Rate

DSPI: Disposable Personal Income

GARCH: generalized autoregressive conditional heteroskedasticity model can be used to capture autocorrelation of the time variant feature of market volatility.

GBM: Gradient boosting machine is another decision tree–based ensemble method. Each tree is a weak estimator trying to estimate the residual error that the estimation of previous trees has caused. Gradually, with a sufficient number of decision trees, the estimation error will decline to a very low level. Unlike Random Forests models which use parallel trees to predict in aggregate, GBM is a sequential tree model with the final prediction as the sum of predictions of all sequential trees.

GFDEGDQ188S: Federal Debt: Total Public Debt as Percent of Gross Domestic Product

GPDI: Gross Private Domestic Investment

gt_covid: Google trend index: COVID

gt_inflation: Google trend index: inflation

gt_interest_rate: Google trend index: interest rate

gt_job: Google trend index: job

gt_market_crash: Google trend index: market crash

gt_pandemic: Google trend index: Pandemic

gt_stock_market: Google trend index: stock market

gt_ukraine: Google trend index: Ukraine

ma_credit: Free Credit Balances in Customers' Securities Margin Accounts

ma_debit: Debit Balances in Customers' Securities Margin Accounts

MTSDS133FMS: Federal Surplus or Deficit

PCE: Personal Consumption Expenditures

PSAVERT: Personal Saving Rate

Random Forests: a random version of the CART models. Multiple subsets are sampled from the training dataset and each subset is used to build a CART model. Explanatory variables are sampled as well so that the relationship between the response variable and the explanatory variables will not be dominated by the most important ones. Less important explanatory variables can contribute to the final prediction as well.

RH_AUC: Assets under custody of Robinhood

RH_MAU: Monthly active users of Robinhood

RVoIV: realized volatility of implied volatility

RVoV: realized volatility of volatility

STL: Seasonality and trend analysis using Loess to decompose a time series into seasonality, trend, and residuals.

SV: stochastic volatility

SVJ: stochastic volatility with jumps

T10YIE: 10-Year Breakeven Inflation Rate

T5YIE: 5-Year Breakeven Inflation Rate

UMCSENT: ty of Michigan: Consumer Sentiment

VAR: vector autoregressive model. It is used to describe the relationship of the modeled variables based on this historical data. By incorporating lagging variables into the analysis through VAR, relationships among leading, coincident, and lagging variables can be better reflected.

VIXCLS: the CBOE volatility index that represents the implied volatility in the S&P 500 Index options over the next 12 months.

VVIX: a widely used measure of volatility of volatility developed by CBOE. It measures the expected volatility of the 30-day forward price of VIX and is calculated using the price of at-money and out-of-money VIX options.

W994RC1Q027SBEA: Net lending or net borrowing: Private

Executive Summary

During the recent pandemic, the global economy and capital market experienced higher volatilities. Although the pandemic and the resulting recession is behind us now, capital markets are still experiencing higher than normal volatilities. Analyzing the market volatility behavior in the context of historical extreme events is helpful for understanding its extremity in terms of not only the level but also the duration.

In general, the U.S. public equity market volatilities during the post-2020 period are still less than those during the Great Depression but are commensurate with historical extreme events such as the 2008 financial crisis and 1987 Black Monday. Some measures such as volatility of volatility are higher than the 2008 financial crisis even though the economic recession period was much shorter during the recent pandemic. High volatilities also stayed for longer during the pandemic, with the degree of volatility clustering similar to that observed during other historical extreme events. Temporal relationships also changed with an example of much higher correlation between implied volatilities and previous days' S&P 500 index returns. Although the post-2020 period may not have created a new level of extremity, it does indicate that the frequency of extreme events may be higher than those suggested by assumptions used in risk management and it is possible that the higher-than-normal volatility may stay.

To understand the drivers of the market volatility, a comprehensive attribution analysis has been performed to identify meaningful relationships between market volatility and potential causes. Economic data, asset data that reflects different investment styles, retail investor data, and event data are used as explanatory variables. In addition, investment related Reddit comments are used to generate summary variables such as sentiment and key word frequency to capture the features of retail investors. Using multiple regression model types, a couple of tree-based models including Random Forests and Gradient Boosting Machine (GBM) are found to be able to explain more than 94% of variation in the market volatility. Random Forests use random data subsets and feature subsets to develop multiple mini models to vote for the best estimate. GBM models also use random data subsets and feature subsets, but in a sequential rather than parallel way. The four categories of explainable variables all showed material contributions to predicting the market volatility, as shown in Table E.1. For example, as leading indicators, the retail investor data explained more than 9% of future market volatility in both model types.

Table E.1
CONTRIBUTION BY DATA CATEGORY

Model Type	Economic Data	Event Data	Investment style data	Retail investor data
Random Forest	20.8%	23.5%	43.1%	12.7%
GBM	20.4%	24.5%	45.7%	9.3%

To make sure that the attribution analysis is examining potential cause-and-effect relationships, explanatory variables are used to predict then-future market volatility, rather than the concurrent volatility. In aggregate, the explanatory variables have both short-term influence and longer-term impact, as shown in Table E.2. For example, in aggregate, the selected explanatory variables explain 39% of the market volatility in one month. They explain about 18.9% of the market volatility in the next trading day.

Table E.2

CONTRIBUTION OF EXPLANATORY VARIABLES BY TIME LAG

Model Type	1-month lag	2-week Lag	1-week lag	3-day lag	1-day lag
Random Forest	39.0%	17.9%	11.4%	12.8%	18.9%
GBM	50.0%	6.3%	5.9%	6.7%	31.2%

The calibrated models can be used as a basis to justify changes in volatility modeling such as the volatility level, the degree of volatility clustering, and relationships. With forward-looking views on how different categories of explanatory variables will evolve, the magnitude of necessary changes in volatility assumptions can be quantified.

An asset allocation example is used to illustrate the potential impact of increased volatility assumption, adopting stochastic volatility models, reflecting volatility clustering, and return jumps. It shows that the optimal equity allocation may be reduced by the consideration of volatility clustering and return jumps, even though the general market volatility stays at the same level. In addition to investment optimization, the suggested changes in volatility modeling are likely to affect other areas such as liability valuation, capital management, hedging, product design, and risk mitigation.

This research contributes to existing literature in three ways. Firstly, different from most research on the same topic, this research is not confined to understand the impact of individual factors such as monetary policies or retail investors on market volatility, but rather a more holistic view of many potential factors at the same time. Secondly, it uses a pure data-driven approach to quantify the impact of different factors with predictive modeling. Given the satisfactory level of prediction accuracy, it can be used as an objective method to determine a future volatility level based on assumptions of the explanatory variables. Lastly, it makes the documented R programs used to perform the analysis publicly accessible for education purpose. The open-source codes are hosted at https://github.com/Society-of-actuaries-research-institute/FP104-Market-Volatility-Risk-in-an-Era-of-Extreme-Events.

Section 1: Introduction

The capital market exhibited a high level of volatility in the recent pandemic event, not only for the entire market, but also some individual assets that are targets of speculation. Coupled with the increasing role of central banks in the capital market through powerful monetary policies, it brings questions on the possibility of structural changes to the market, a new norm of market volatility, and the potential need to reposition our exposures to different asset classes based on our financial obligations and risk appetite.

Literature exists to explain the recent capital market movements from different angles. Mazur et al. (2020) investigated the negative correlation between extreme asymmetric volatility and stock returns for loser stocks. Bansal (2020) explained the excessive volatility and the unshaken confidence of financial institutions from a behavioral finance lens and discuss some cognitive errors and biases relevant during and after the crisis. Onali (2020) investigated the impact of COVID-19 cases and related deaths on the stock market and its volatility level. Capelle-Blancard and Desroziers (2020) assessed how stock markets have integrated public information about the COVID-19, the subsequent lockdowns, and the policy reactions.

Leveraging on existing literature, this report wants to provide a comprehensive explanation of what happened recently in the capital market, what caused the structural changes, if any, and more importantly, the implications on our actuarial assumptions, investment, hedging, and risk management strategies. In this report, market volatility is considered as a general term that contains not only the standard deviation of market data, but also other risk measures such as VaR and CTE. In addition to the level of market volatility, nonlinear relationships among asset classes also have material impacts and are studied as well. While this research introduces popular theories and stories of structural changes in the capital market due to the pandemic, it adopts the data-driven approach to avoid subjective judgement as much as possible.

We proceed as follows:

- Section 2 (Key Implications) provides a high-level overview of the potential implications of the quantitative analysis of historical data and the attribution analysis detailed in the next few sections.
- Section 3 (Historical Market Volatility Analysis) analyzes recent capital market data in terms of market volatility and relationships and assesses them against observations in historical extreme events. Volatility clustering, discrete jumps, and nonlinear relationships, both contemporary and temporal, are also studied to identify possible structural changes of the capital market. Key findings are listed at the end of each subsection.
- Section 4 (Attribution Analysis) uses predictive modeling to determine the influential factors on market volatility. Factors considered include pandemic related data, monetary policies, fiscal policies, investor behaviors and idiosyncratic factors. To be able to have an in-depth analysis of investor behaviors, Reddit data which includes all public comments from January 2006 to June 2021 is also used to gauge the influence of retail investors and social media. For the identified key factors, efforts were made to evaluate whether these factors will be in effect in the long term, and whether they indicate a permanent structural change of the capital market. Key findings are listed at the end of each subsection.
- Section 5 (Actuarial and Investment Implications) focus on the financial impacts of identified new assumptions and patterns of market volatility in a variety of areas, including economic assumptions, asset allocation, hedging strategy, and risk management. This section starts with general discussions on those areas, followed by a case study using a sample life insurance portfolio with embedded options to illustrate the quantification process. Key findings are listed at the end of each subsection.
- Section 6 (Further Developments) discusses potential extensions of this research.

- Section 7 (Conclusion) summarizes the key points of this research and concludes the main body of the report.
- Appendix A (Market Volatility Analysis) provides more details and supplementary analysis to support Section 3.
- Appendix B (Reddit Data Analysis) provides additional information about the analysis performed on the Reddit data. It includes data cleaning and natural language processing (NLP) models.
- Appendix C (Feature Importance) describes the methods used to determine the importance of individual explanatory variables in the attribution analysis.
- Appendix D (Open-Source Program) describes the programs built for this research that are publicly accessible.

Section 2: Key Implications

Given the large number of models used and the amount of analysis performed in this research, this section provides a summary of potential implications of the research, with supporting details in later sections.

Market Volatility Level

The post-2020 period exhibited extreme market movements at similar levels with the 1987 Black Monday and 2008 financial crisis, but less extreme than the Great Depression. Specifically, the implied volatility and volatility of volatility, two measures that are linked to hedging cost, exhibited a high level of implied volatility compared to the 2008 financial crisis. This may lead to an adjustment of the market volatility assumption if it is believed that factors contributing to the market volatility in the post-2020 period will persist to a certain degree. Based on the attribution analysis, economic data, event data, investment style data, and retail investor data showed high prediction power on short-term market volatility.

Supporting Analysis: Section 3.1 Market Volatility and Section 4 Attribution Analysis

Modeling Frequency

The post-2020 period showed strong volatility clustering that may justify higher modeling frequency. Modeling frequency can have a significant impact on results of financial projection. Models with low frequency such as annual and quarterly frequency may underestimate the risk exposure significantly and the short-term impact of extreme events. A higher frequency also gives the flexibility to evaluate risks at a lower frequency without losing the important details. For example, daily equity index returns can be transformed to monthly, quarterly, or annual returns.

Supporting Analysis: Section 3.2 Volatility Clustering

Volatility term structure

Models with constant volatility or fixed volatility term structure may be replaced with stochastic volatility models that reflect volatility of volatility and volatility clustering.

Supporting Analysis: Volatility of volatility analysis in Section 3.1 Market Volatility, and Section 3.2 Volatility Clustering

Significance of outliers

When outliers cannot be explained by stochastic volatility models, jump diffusion models that contain discrete jumps may be used to reflect the extreme events.

Supporting Analysis: Section 3.3 Jumps.

Number of scenarios used in stochastic analysis

The sampling errors increase with a higher volatility level, a higher degree of volatility clustering and/or the existence of discrete jumps. A larger set of scenarios may be needed to maintain the same level of convergence when calculating risk measures such as value at risk and tail value at risk using real-world scenarios, or even the fair market value of liability cashflows with embedded options and guarantees using risk-neutral scenarios, if the magnitude of volatility increase is material.

Supporting Analysis: Section 3.1 Market Volatility and Section 3.2 Volatility Clustering

Nonlinear contemporary relationships

With higher correlations observed in most extreme events, nonlinear relationships need to be reflected through methods such as state-dependent correlation matrices or copulas.

Supporting Analysis: Section 3.4.1 Contemporary Relationship

Temporal relationships

When using models with high frequency, temporal relationships also need to be incorporated in the models through autocorrelations and cross correlations.

Supporting Analysis: Section 3.4.2 Temporal Relationship and Section 4. Attribution Analysis

Changes in the assumptions or modeling approaches discussed above may lead to different modeling results and therefore different optimal strategies.

1. **Liability-driven investment strategy**. The impact of volatility assumption and modeling approaches can have material impact on the resulting optimal asset allocation plan. Increasing the equity volatility assumption causes reduction in optimal equity allocation. Applying stochastic volatility reduces the optimal equity allocation in many cases, even though the general market volatility level stays the same. Adding return jumps also has some marginal impact on equity allocations, as shown in the numerical example in *Section 5.2 Asset Allocation Optimization*.

2. **Liability valuation and capital requirement**. The distribution of financial outcomes is likely to have larger ranges, and more importantly, heavier tails, given that the assets backing liability may be more volatile and the underlying assets that the guaranteed liability value is linked to may be more volatile. For liability valuation and capital management that use high confidence levels, increases in the liability values and capital requirements are expected, ceteris paribus.

3. **Hedging strategies** that focus on first-order sensitivities such as Delta (sensitivity to equity) and Rho (sensitivity to interest rate) may see lower hedging effectiveness. In addition, increasing hedging costs during extreme events may make certain hedging programs too expensive to implement. Second-order sensitivities such as Gamma (sensitivity to Delta) and Vega (sensitivity to implied volatility) may need to be incorporated into hedging programs to be immune to stochastic volatilities. Dynamic hedging programs need to monitor these second-order sensitivities and adjust hedging positions in a timely manner. Financial derivatives on market implied volatility such as volatility swaps and options may be used more frequently to mitigate the risk of having volatile cost of first-order hedging. Hedging positions may be assessed and adjusted at least on a daily basis to reduce the impact of market illiquidity during extreme events.

4. The **cost of providing guarantees of investment performance** may be found too high. The guaranteed level may be lowered together with lower premium rates or higher upside potential. For guarantees that are backed with long-term asset liability matching strategies, appropriate penalty for early termination may be designed to offset the cost of asset and liability mismatch. Effective communication with policyholders is also important to manage their expectation and behaviors to mitigate the exposure to heightened volatility risk.

5. The **risk-absorbing capability** may be reassessed given new volatility assumptions. Investment risk may be shared with the capital market using reinsurance, structured instruments, and financial products that the payments are linked with capital adequacy ratio.

Even though volatility assumptions may not be changed immediately, given the possibility that recent market volatility behaviors may have long-term impact based on the findings from the attribution analysis, it is beneficial to quantify the potential financial impact if volatility assumptions and models change and make contingent plans that may be triggered if market conditions such as conditional volatility reaches a certain threshold.

Section 3: Historical Market Volatility Behavior

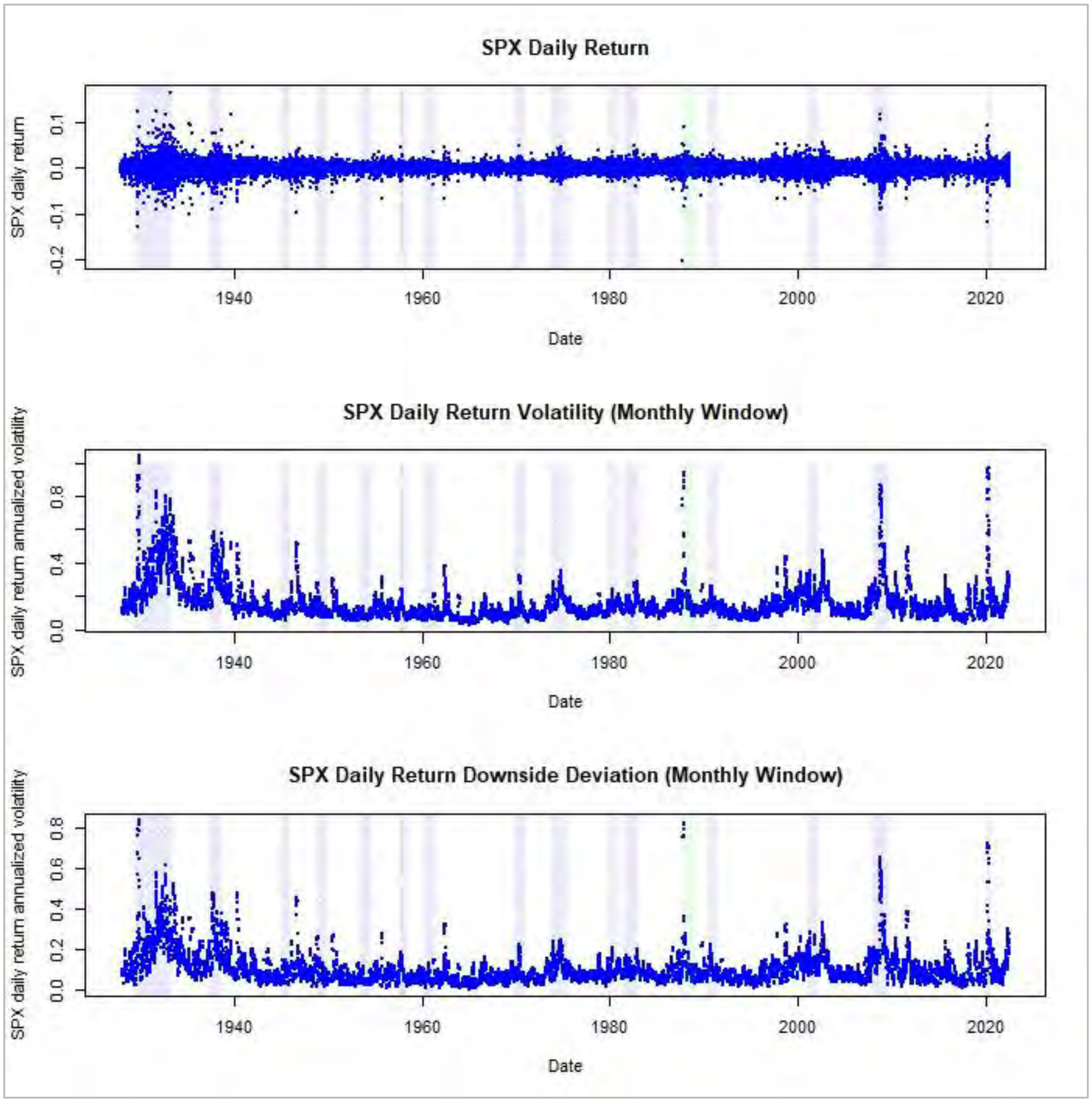
Since the outbreak of the COVID pandemic in early 2020, capital markets around the world have been experiencing volatilities at a level not seen after the 2008 financial crisis. To understand its implications, this section analyzes the recent market volatility in the context of the nearly 100-year history of the U.S. capital market.

3.1 MARKET VOLATILITY

The market volatility level in the post-2020 period matches the level observed in extreme events including the great depression (August 1929 – March 1933), 1987 Black Monday (October 1987), and the 2008 financial crisis (December 2007 – June 2009). Figure 1 shows the S&P 500 index (SPX) daily returns, return volatilities, and downside deviation from January 1928 to June 2022.

Figure 1

SPX DAILY RETURNS AND ANNUALIZED RETURN VOLATILITIES



Notes:

1. *SPX daily returns: Yahoo! Finance*
2. *Blue shaded areas: economic recessions compiled by the National Bureau of Economic Research (NBER).*
3. *Green shaded area: 1987 Black Monday capital market crash that took two years to recover. However, it did not lead to an economic recession.*
4. *Daily return volatility is annualized by multiplying it with* $\sqrt{252}$.
5. *Downside deviation is calculated using a minimum acceptable return of zero.*

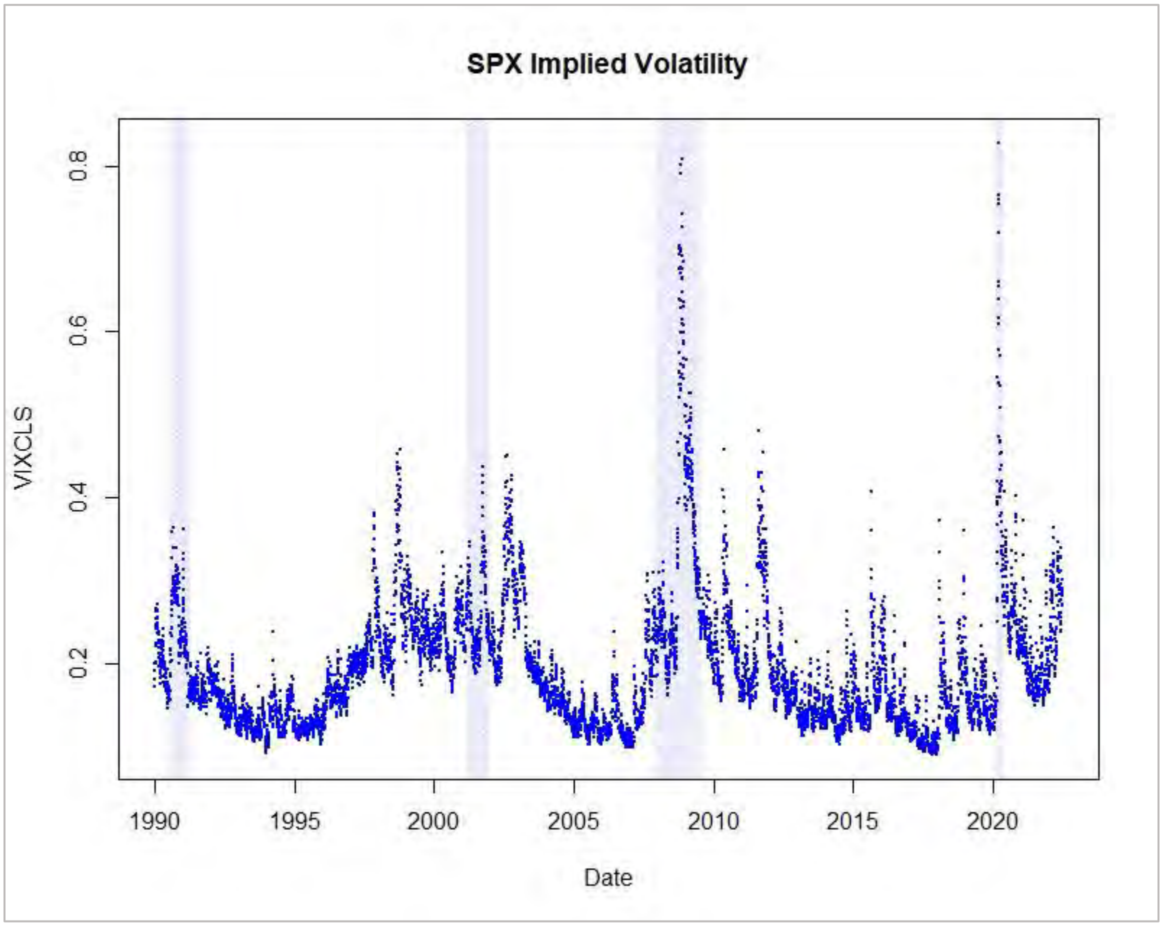
Based on the realized market volatility, the post-2020 period exhibited extreme market movements at similar levels with the 1987 Black Monday and the 2008 financial crisis, but less extreme than the Great Depression. The downside deviation shows a similar story. The descriptive statistics listed in Table 1 also show the extremity of the recent period, but still comparable with other extreme periods.

Table 1
DESCRIPTIVE STATISTICS OF SPX DAILY RETURNS

	All Periods (Jan 1928 — Apr 2022)	Great Depression (Aug 1929 – Mar 1933)	Black Monday (Oct 1987 – Dec 1988)	Financial Crisis (Dec 2007 – Jun 2009)	COVID Pandemic (Jan 2020 — June 2022)
No. of records	23683	910	317	397	629
Min	-20.5%	-12.9%	-20.5%	-9.0%	-12.0%
Max	16.6%	16.6%	9.1%	11.6%	9.4%
Mean	0.0%	-0.1%	0.0%	-0.1%	0.0%
Standard Deviation	1.2%	2.8%	1.9%	2.4%	1.6%
Skewness	-0.12	0.51	-3.96	0.16	-0.57
Kurtosis	20.50	6.79	47.24	6.53	14.11
VaR 1%	-3.4%	-7.3%	-5.0%	-6.7%	-4.4%
Left-Side TVaR 1%	-5.1%	-8.9%	-10.2%	-8.6%	-7.1%
VaR 99%	3.3%	8.3%	3.6%	6.5%	4.5%
Right-Side TVaR 99%	5.1%	10.8%	5.7%	9.1%	6.8%

The realized market volatility is useful for investors who plan to hold the portfolio for a while. For investors who want to mitigate the risk of uncertain market volatility immediately through financial derivatives, implied volatility is another important measure to study. Implied volatility is the volatility parameter(s) used in an option pricing model, such as the famous Black Scholes Merton model. The value of the implied volatility is determined such that the theoretical option price based on the pricing model is the same as the market price of the option. Higher implied volatilities lead to higher option prices, and therefore higher costs of risk mitigation and hedging. Figure 2 shows the CBOE volatility index (VIXCLS) that represents the implied volatility in the S&P 500 Index options over the next 12 months. The post-2020 period exhibited a high level of implied volatility compared to the 2008 financial crisis.

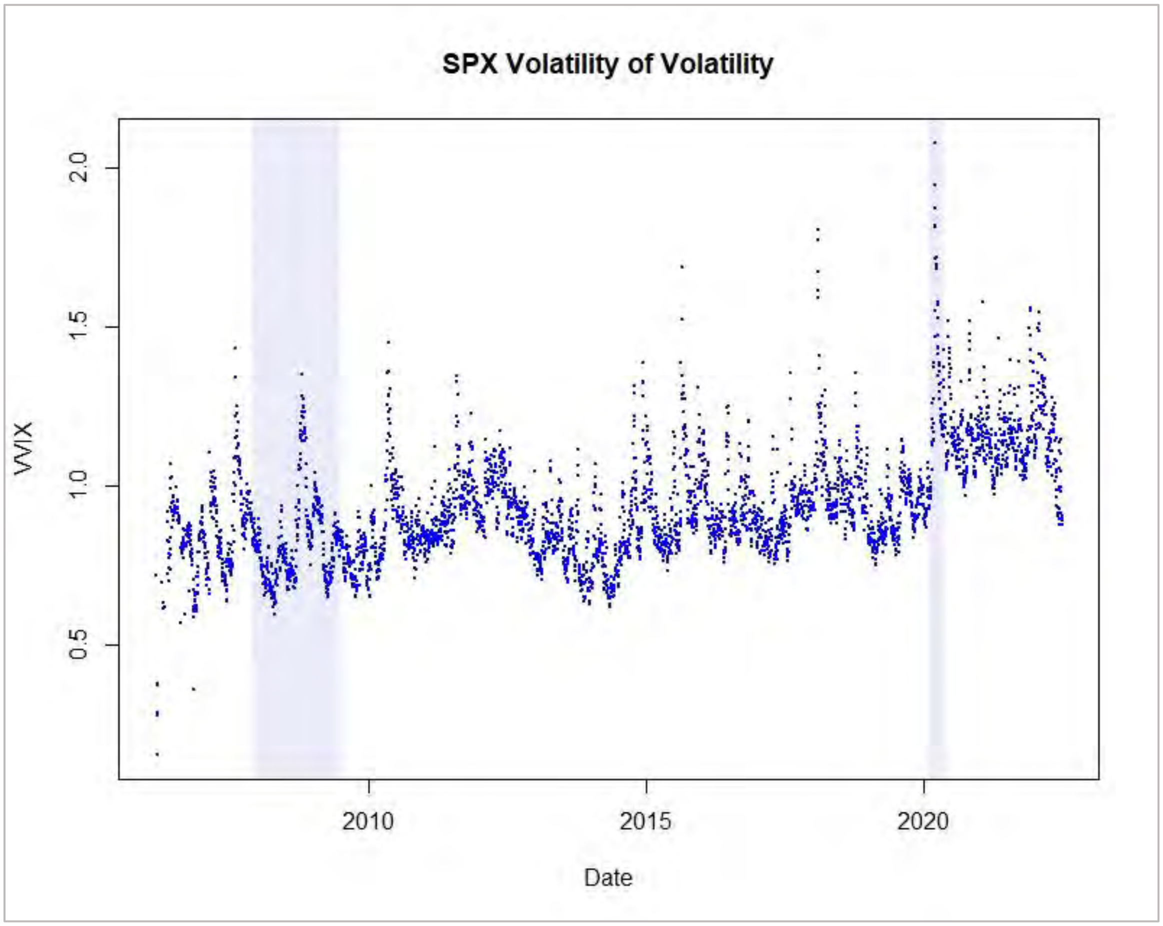
Figure 2
SPX IMPLIED VOLATILITIES



Data Source: FRED Economic Database

In addition, volatility of volatility is another popular measure of market conditions. Many investors and risk managers use VIX options to manage the volatility risk, with both put and call options on VIX, the implied volatility index. Volatility of volatility is also important to understand the fluctuation of hedging costs when volatility options are used. Figure 3 shows the CBOE VVIX, a widely used measure of volatility of volatility. It measures the expected volatility of the 30-day forward price of VIX and is calculated using the price of at-money and out-of-money VIX options. A VVIX of 100% means the expected volatility (standard deviation) is the current 30-day forward price of VIX. The VVIX can be interpreted as the volatility of VIX futures rather than the volatility of VIX. However, VIX futures track the VIX closely in a timely manner.

Figure 3
VOLATILITY OF VIX OPTIONS (MARCH 2006 – JUNE 2022)

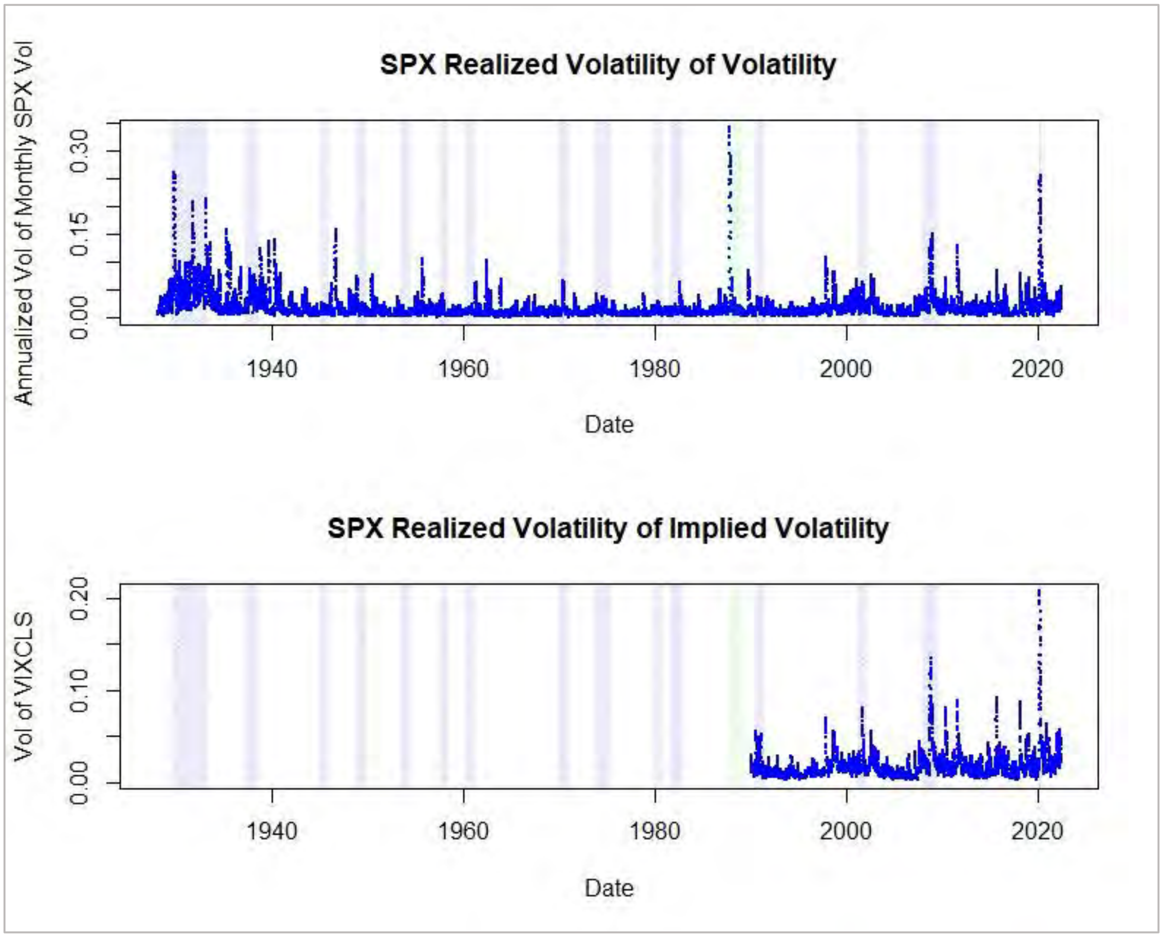


Data source: CBOE VVIX historical data (https://cdn.cboe.com/api/global/us_indices/daily_prices/VVIX_History.csv)

During the post-2020 pandemic period, the VVIX is much higher than that during the 2008 financial crisis. As the VVIX started from early 2006, to be able to view the volatility of volatility in a longer historical period, two additional measures are constructed: realized volatility of volatility (RVoV) and realized volatility of implied volatility (RVoIV).

To calculate RVoV, the first step is to calculate the volatility of SPX daily returns using a 21-day window. The RVoV is then calculated as the volatility of the SPX return volatility calculated in the previous step, using 21-day window as well. The 21-day window is chosen to represent monthly experience because each month has approximately 21 trading days. On the other hand, the RVoIV is calculated as the volatility of the implied volatility VIXCLS using a 21-day window as well. Figure 4 shows the RVoV from 1928 to 2022 and the RVoIV from 1990 to 2022. Both measures indicate that the post-2020 period has the highest volatility of volatility in history except during the Great Depression and the 1987 Black Monday market crash.

Figure 4
REALIZED VOLATILITY OF VOLATILITIES



In Appendix A, similar analysis is performed for NASDAQ and Russell 2000 index returns, with similar observations to those made using the S&P 500 data. However, the technology sector and the small-cap equity markets are more volatile than the large-cap equity markets during the recent period, when comparing to the 2008 financial crisis. The pandemic hit the small businesses harder and boosted the technology sector, which is different from the financial crisis. During the post-2020 pandemic period, the VVIX is much higher than that during the 2008 financial crisis.

Key Findings

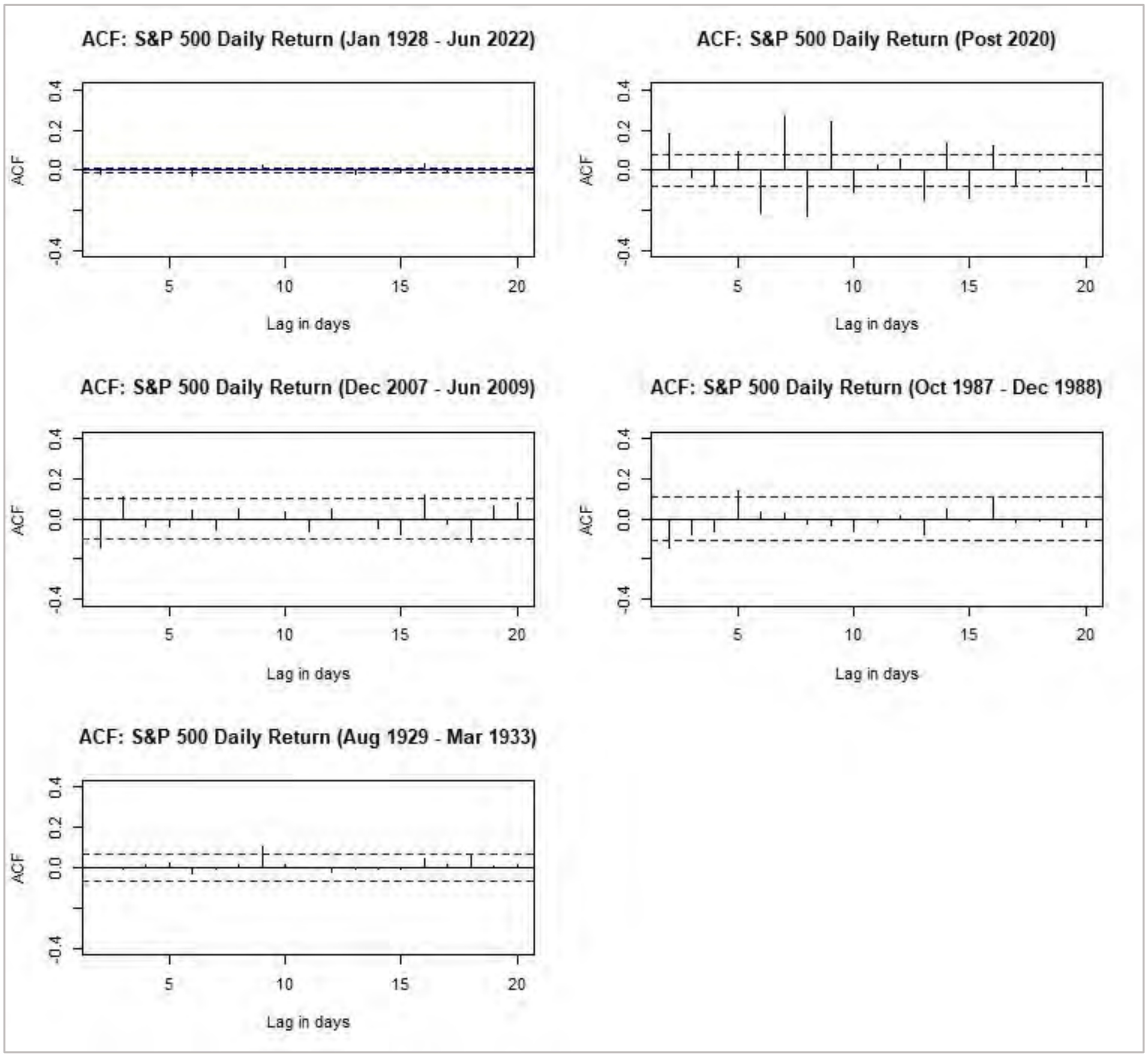
1. The post-2020 period exhibited extreme market movements at similar levels with the 1987 Black Monday and 2008 financial crisis, but less extreme than the Great Depression.
2. Implied volatility is the volatility parameter(s) used in an option pricing model, such as the famous Black Scholes Merton model. Higher implied volatilities lead to higher option prices, and therefore higher costs of risk mitigation and hedging. The post-2020 period exhibited a high level of implied volatility compared to the 2008 financial crisis.
3. Volatility of volatility is another popular measure of market conditions. Many investors and risk managers use VIX options to manage the volatility risk, with both put and call options on VIX, the implied volatility index. Volatility of volatility is also important to understand the fluctuation of hedging costs when volatility options are used. During the post-2020 pandemic period, the VVIX is much higher than that during the 2008 financial crisis.

3.2 VOLATILITY CLUSTERING

In addition to the level of volatilities examined in the previous section, volatility clustering can also have material impacts on risk exposure measured with a time horizon longer than one day. Volatility clustering is a phenomenon that large market changes are likely to be followed by large changes no matter the direction, while small changes are usually followed by small changes. A period of consistent high volatility can lead to higher cost of risk mitigation, breach of risk tolerance, and potentially insolvency.

Figure 5 compares the autocorrelation function (ACF) of daily returns with a maximum lag of 20. The autocorrelations are minimal considering all the available data from January 1928 to April 2022. The autocorrelations during the post-2020 period is much higher compared to other extreme periods including the Great Depression, 1987 Black Monday, and the 2008 financial crisis.

Figure 5
ACF OF S&P 500 DAILY INDEX RETURNS



Note: the dotted lines represent the threshold beyond which the autocorrelations are statistically different from zero.

To evaluate the degree of volatility clustering, time series models such as autoregressive moving average (ARMA) and generalized autoregressive conditional heteroskedasticity (GARCH) can be used to capture autocorrelation of the time variant feature of equity volatility. ARMA-GARCH models are used to analyze historical S&P 500 index daily returns.

$$ARMA(p,q) \sim r_t = c + \varepsilon_t + \sum_{i=1}^{p} \varphi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

$$\varepsilon_t = z_t \sigma_t$$

$$GARCH(p,q) \sim \sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

Where

r_t = S&P 500 index daily return. It is calculated as $\frac{S_t}{S_{t-1}} - 1$.

z_t = i.i.d. with zero mean and unit variance.

The distribution of z_t should be chosen according to the experience data. Table 1 showed that the historical daily returns exhibit skewness and heavy tails that cannot be captured by Gaussian distributions. In this example, z_t is assumed to follow the skewed generalized error distribution (SGED). It has the following probability density function:

$$f_{SGED}(x; \mu, \sigma, \lambda, p) = \frac{pe^{-\left\{\frac{|x-\mu+m|}{v\sigma[1+\lambda sign(x-\mu+m)]}\right\}^p}}{2v\sigma\Gamma(1/p)}$$

Where

μ = location parameter. It is zero for z_t.

σ = scale parameter. It is one for z_t.

λ = skewness parameter.

p = shape parameter.

$m = \frac{2^{\frac{2}{p}} v\sigma\lambda\Gamma\left(0.5 + \frac{1}{p}\right)}{\sqrt{\pi}}$ if the mean of variable x equals μ.

$v = \sqrt{\frac{\pi\Gamma\left(\frac{1}{p}\right)}{\pi(1 + 3\lambda^2)\Gamma\left(\frac{3}{p}\right) - 16^{\frac{1}{p}}\lambda^2\Gamma\left(0.5 + \frac{1}{p}\right)^2\Gamma\left(\frac{1}{p}\right)}}$ if the volatility of variable x equals σ.

To facilitate comparison, ARMA(1,1) and GARCH(1,1) with the SGED are used to analyze historical S&P 500 daily index returns for different time periods. ARMA and GARCH models are fitted jointly to the historical data using the method of maximum likelihood estimation. Other values of parameters of *p* and *q* are tested without observing material improvement, if any, of loglikelihood. Table 2 lists the parameters of the fitted models.

Table 2
ARMA + GARCH MODEL PARAMETERS

Parameter	All Periods (Jan 1928 — Jun 2022)	Great Depression (Aug 1929 – Mar 1933)	Black Monday (Oct 1987 – Dec 1988)	Financial Crisis (Dec 2007 – Jun 2009)	COVID Pandemic (Jan 2020 — Jun 2022)
$\boldsymbol{c}$	0.000	0.000	0.000	0.000	0.000
$\boldsymbol{\varphi_1}$	0.202	-0.540	0.229	0.512	0.722
$\boldsymbol{\theta_1}$	-0.164	0.463	-0.340	-0.661	-0.833
$\boldsymbol{\omega}$	0.000	0.000	0.000	0.000	0.000
$\boldsymbol{\alpha_1}$	0.091	0.178	0.027	0.112	0.180
$\boldsymbol{\beta_1}$	0.905	0.819	0.950	0.880	0.808
$\boldsymbol{\mu}$	0	0	0	0	0
$\boldsymbol{\sigma}$	1	1	1	1	1
$\boldsymbol{\lambda}$	0.945	1.025	0.922	0.874	0.664
$\boldsymbol{p}$	1.301	1.337	1.000	1.508	1.592

Based on the 1-day autocorrelation ($\boldsymbol{\varphi_1}$) of daily index return, the post-2020 period has a high positive correlation compared to other study periods. The post-2020 period also has high autocorrelation of the conditional volatilities ($\boldsymbol{\beta_1}$), similar to other extreme periods. This is lower than the value using all periods data, as observed in other extreme periods as well. This is because the higher autocorrelation of daily index returns has captured part of the clustering impact.

Standardized residuals from the ARMA and GARCH models are compared to standard Normal distribution and fitted SGED to understand how well heavy tails have been captured by using the SGED. Figure 6 draws the quantile-quantile (Q-Q) plots between empirical distribution and theoretical distributions. The SGED does a better job capturing both the left and right heavy tails than the Normal distribution.

Figure 6
SPX RETURN RESIDUALS Q-Q PLOTS

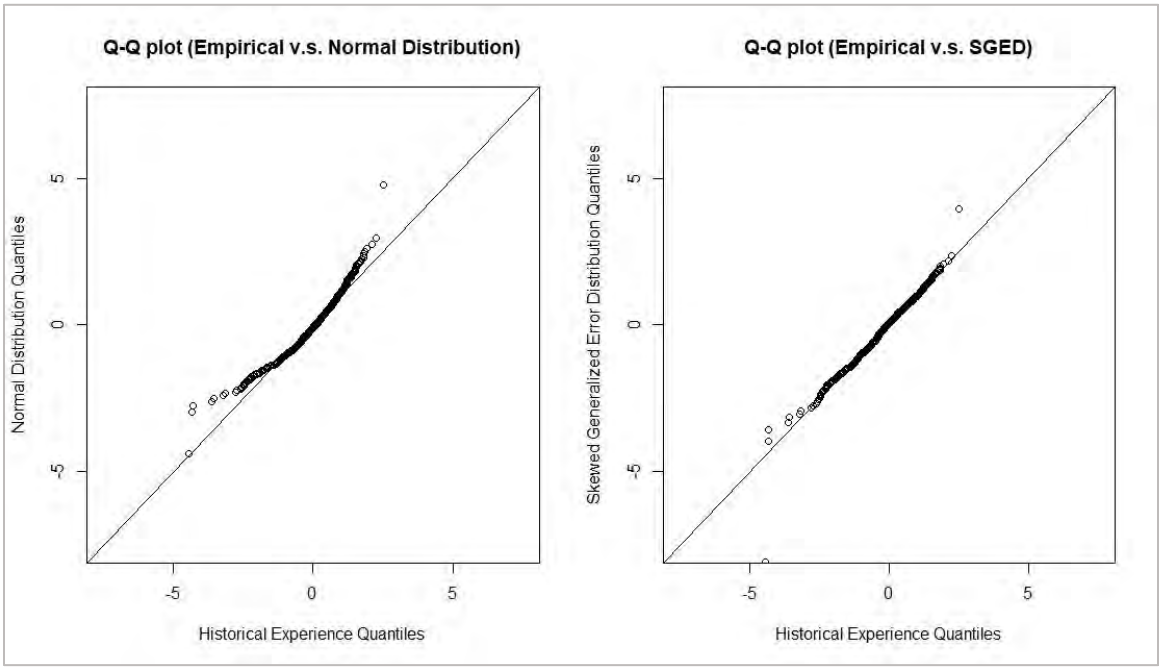


Figure 7 shows the conditional volatilities σ_t estimated by the GARCH model that is calibrated to the data since 1928, together with the actual SPX returns. Condition volatilities are important information for calculating risk measures such as value at risk and tail value at risk based on market conditions. The recent period had higher conditional volatilities than the 2008 financial crisis.

Figure 7
SPX RETURN CONDITIONAL VOLATILITY (ALL PERIODS)

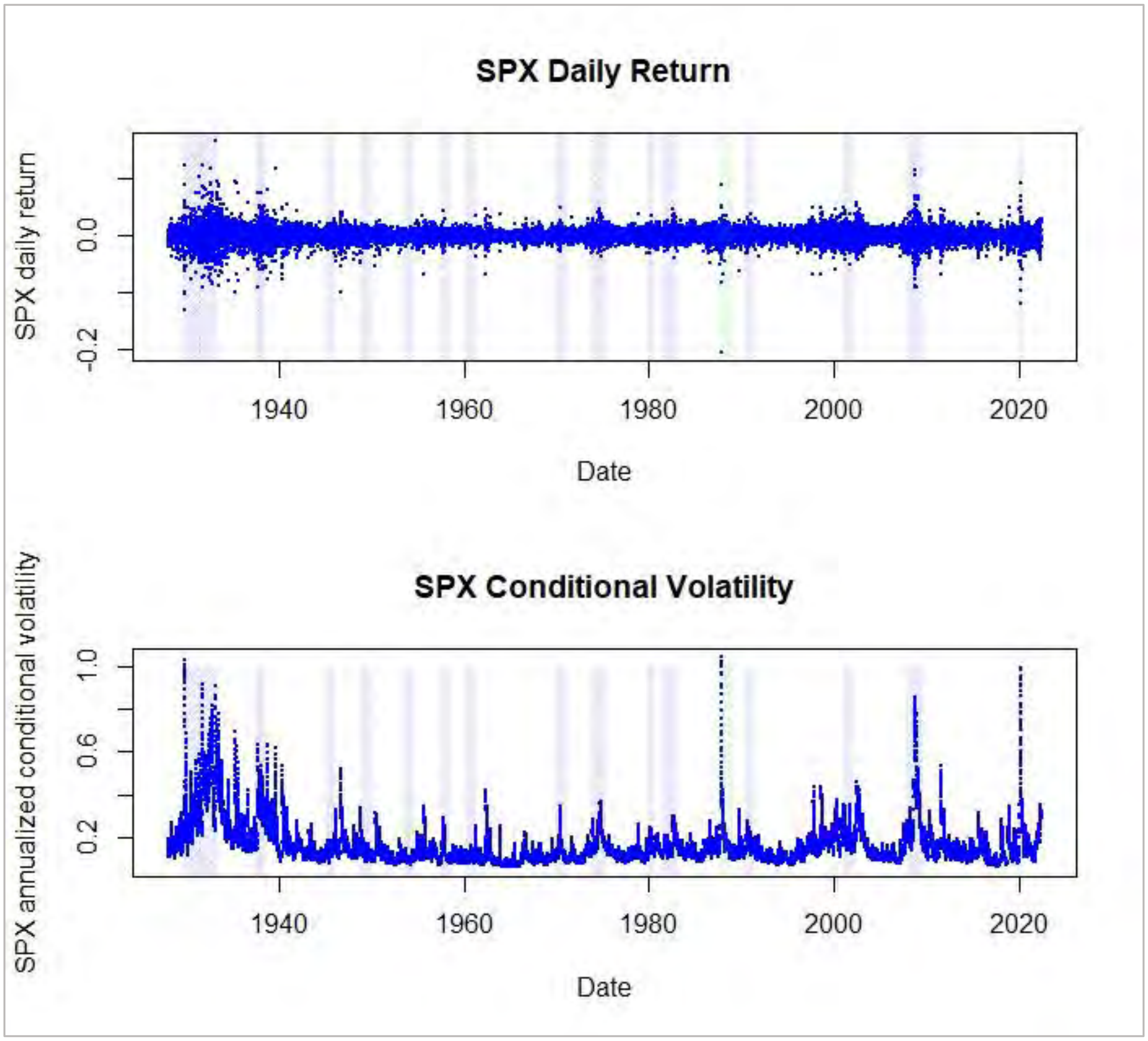
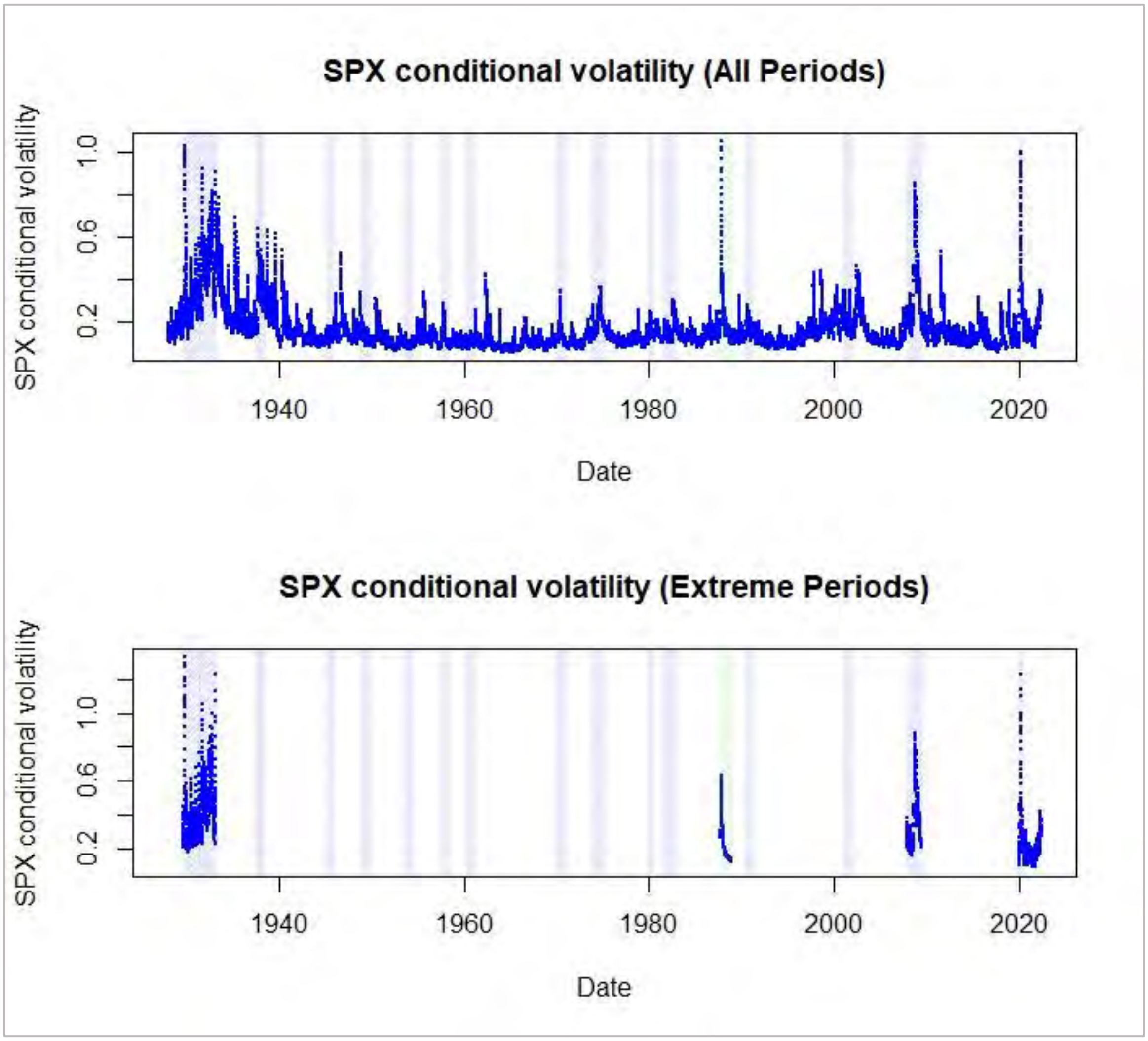


Figure 8 shows the conditional volatilities based on models that are calibrated to all periods on the top and to different extreme periods separately at the bottom. Whether using all available historical data or post-2020 data for model fitting, the level of conditional volatilities is close to the historical record in the recent period. If the recent experience is believed to last in the near future, higher risk measures are expected.

Figure 8
SPX RETURN CONDITIONAL VOLATILITY (EXTREME PERIODS)



Note: extreme periods include Great Depression (Aug 1929 – Mar 1933), Black Monday (Oct 1987 – Dec 1988), Financial Crisis (Dec 2007 – Jun 2009), and COVID Pandemic (Jan 2020 — June 2022).

With the fitted model, future daily VaR can be predicted. Conditional daily VaR can be estimated by simulating future returns using the following method:

for l = 1 to No_of_days:

- Estimate the expected daily return l days after T, the ending date of the historical data.

$$\mathbb{E}(r_{T+l}) = c + \sum_{i=1}^{p} \varphi_i r_{T+l-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{T+l-j}$$

$$\varepsilon_{T+l-j} = 0 \ \ if \ \ l - j > 0$$

- Estimate the expected conditional variance.

$$\sigma_{T+l}^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{T+l-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{T+l-j}^2$$

- Simulate the error term.

  $\varepsilon_{T+l} = z_{T+l}\, \sigma_{T+l}$

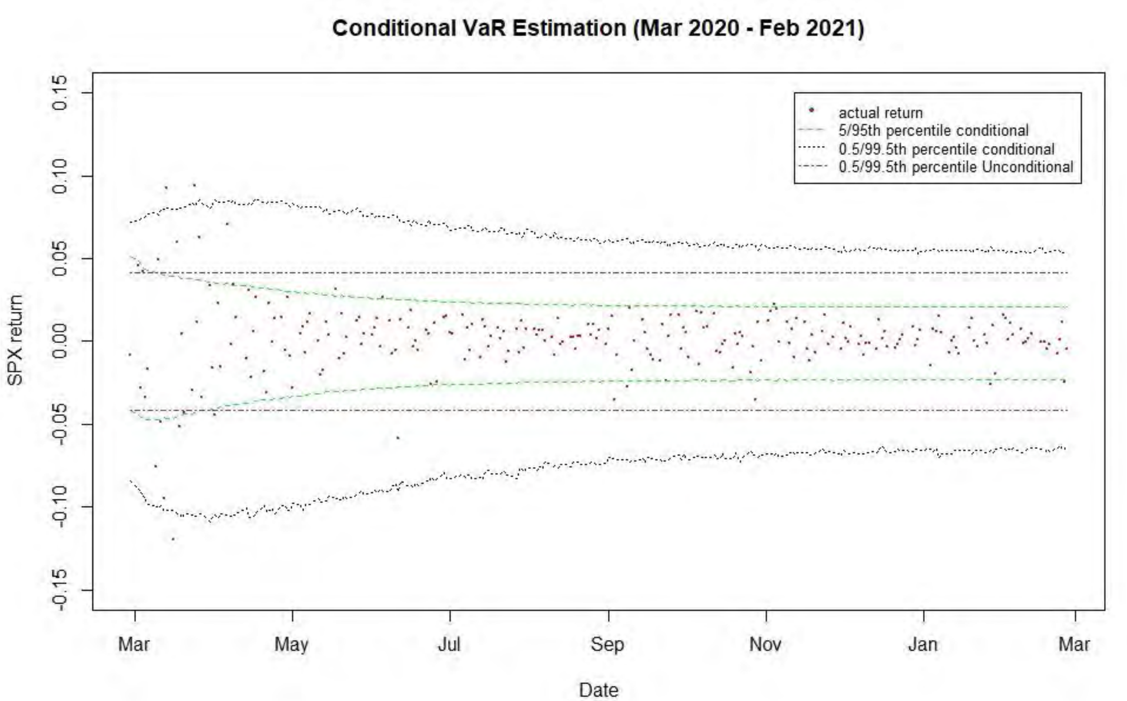
  z_{T+l} is simulated from the SGED.

- Calculate the simulated return.

  $r_{T+l} = \mathbb{E}(r_{T+l}) + \varepsilon_{T+l}$

The process can be repeated many times to simulate multiple paths of returns. Based on the post-2020 period, conditional VaR of the daily returns are predicted and compared with actual returns from March 2020 when the market became very volatile, to February 2021 for an entire year. Figure 9 shows the 90% confidence interval and 99% confidence interval based on conditional simulation using the ARMA and GARCH model, and the 99% confidence interval based on unconditional estimation using the Gaussian distribution. As expected, the conditional VaR estimation can reflect the impact of current volatility level in the projection, while the 99% confidence interval increases in the first month before dampening. It also captures more extreme returns. On the other hand, the 99% confidence interval of unconditional estimation, represented by the two straight lines, may underestimate the extreme returns in this example.

Figure 9
SPX RETURN CONDITIONAL VAR ESTIMATION



For investors with a longer time horizon, the impact of volatility clustering can be material. Usually, historical data is not sufficient to support a statistically credible estimation of annual VaR, in addition to the risk that economic structural changes can make historical data irrelevant. Alternatively, annual VaR can be estimated as the product of daily VaR and $\sqrt{252}$.[1] However, the underlying assumption is that the daily returns are individually independently distributed (i.i.d.). To be able to reflect the autocorrelations of both returns and conditional volatilities, annual VaR can be estimated using the simulation process described above. Under each simulated scenario, the annual return can be constructed by a simulated path of daily returns which reflect autocorrelations. The annual VaR can then be estimated based on simulated annual returns. Table 3 shows the annual VaR at different confidence levels using the conditional and unconditional estimation. The conditional estimation expects a more severe market crash than the unconditional estimation. For the upside potential measured by the 95th and 99.5th percentiles, both estimation methods expect similar results when starting from a volatile and bear market in March 2022 in this example.

Table 3
S&P 500 INDEX RETURN ANNUAL VAR ESTIMATION

Confidence Level	0.5%	5%	95%	99.5%
Conditional Estimation	-0.72	-0.16	0.56	0.79
Unconditional Estimation	-0.50	-0.26	0.57	0.81

Key Findings

1. The autocorrelations of SPX daily returns during the post-2020 period is much higher compared to other extreme periods including the Great Depression, 1987 Black Monday, and the 2008 financial crisis.
2. The level of conditional volatilities during the post-2020 period is close to the historical record in the recent period which indicates a high degree of volatility clustering.
3. Using the ARMA and GARCH model that incorporates volatility clustering, the conditional estimation expects a more severe market crash than the unconditional estimation in terms of annual VaR.

[1] "252" is the average number of trading days in a year.

3.3 JUMPS

The movements of equity index returns may be modeled as the combination of volatilities and jumps. Jumps can be used to model discrete and independent extreme movements. Any changes in the frequency and severity of jumps may also indicate any potential structural changes. To quantify the jump process, the change in equity index S is assumed to follow the following process:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + dP_t$$

$$dP_t = z_t S_t dN_t$$

Where

S_t: equity index value at time t.

W_t: a standard Wiener process.

P_t: a jump diffusion process that is a Poisson process (N_t) where the jump size follows $z_t S_t$.

N_t: a Poisson process with intensity parameter λ.

z_t: a random variable that follows a Gaussian distribution $N(\mu_z, \sigma_z)$.

The Bayesian Monte Carlo Markov Chain (MCMC) method is used to estimate the parameters of the four extreme periods in Table 4. Bayesian MCMC uses a simulation method to gradually adjust model parameters until they converge to stable values. Each model parameter is assigned with an arbitrary prior distribution, usually reflecting a guess of the mean and range of the parameter. However, wrong guesses are fine but may increase the calibration time. An iteration process is then established in which simulation and inference, posterior distributions of model parameters are simulated and updated until they become stable. Both the estimated frequency (λ) and severity ($N(\mu_z, \sigma_z)$) of the jumps during the post-2020 period stayed below the historical maximum level but still significant.

Table 4

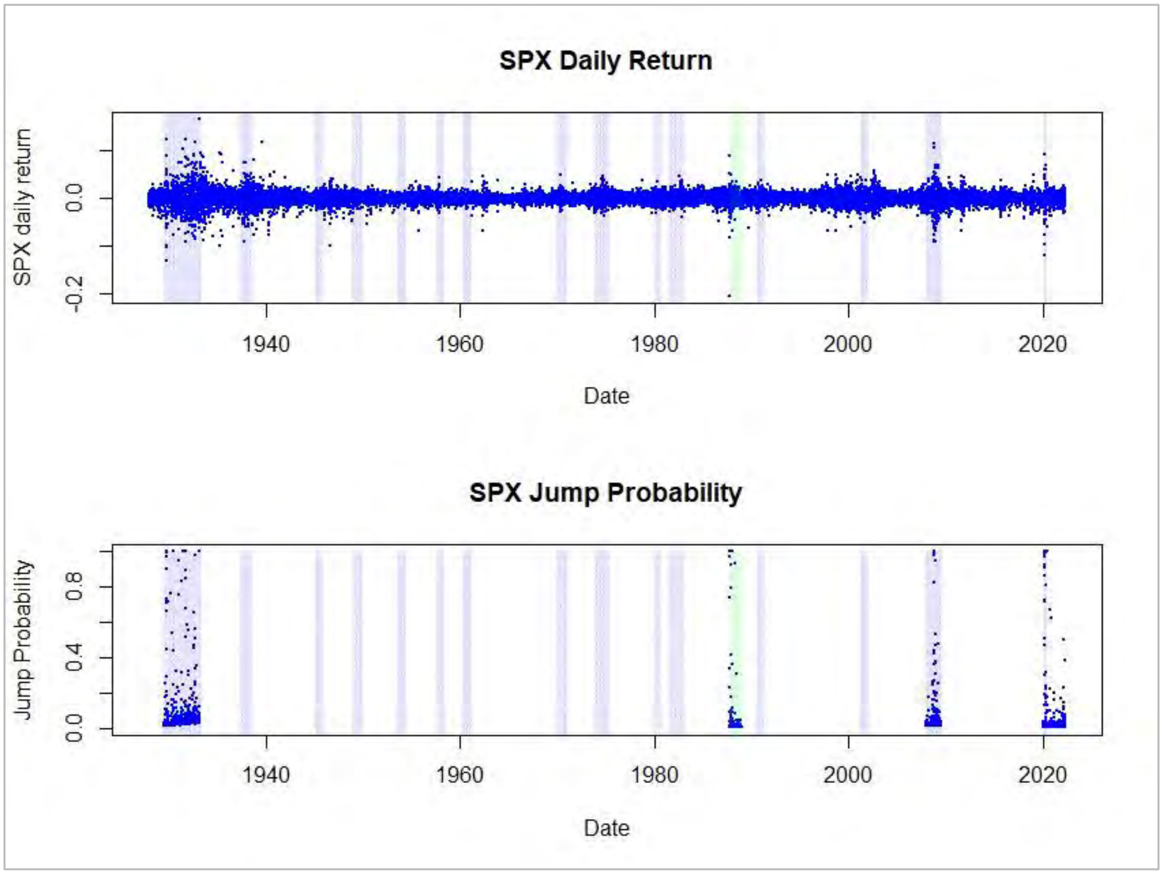
JUMP DIFFUSION MODEL CALIBRATION RESULT

Parameter	Great Depression (Aug 1929 – Mar 1933)	Black Monday (Oct 1987 – Dec 1988)	Financial Crisis (Dec 2007 – Jun 2009)	COVID Pandemic (Jan 2020 — Jun 2022)
μ	-0.397	0.144	-0.237	0.213
σ	0.170	0.126	0.224	0.145
λ	17.594	11.138	13.224	12.814
μ_z	0.005	-0.022	-0.003	-0.006
σ_z	0.062	0.092	0.096	0.057
Jump probability > 0.5	3.6%	3.2%	1.8%	3.4%

Note: the jump probability is estimated by calculating the probability that the stock index value based on the diffusion process ($dS_t = \mu S_t dt + \sigma S_t dW_t$) without jump components will not exceed the actual index value based on historical data. For example, if the actual index return is 0.2, the chances that the simulated returns using μ = 0.06 and σ = 0.2 exceeding 0.2 is $1 - \Phi\left(\frac{0.15-0.06}{0.15}\right) = 0.309$. Here is the cumulative distribution function of the standard Gaussian distribution.

Figure 10 shows the jump probability of the four periods studied. For the post-2020 period, even after the short recession that ended in April 2020, jumps are not rare.

Figure 10
JUMP PROBABILTY OF S&P 500 DAILY INDEX RETURNS



Key Findings

The movements of equity index returns may be modeled as the combination of volatilities and jumps. Jumps can be used to model discrete and independent extreme movements. For the post-2020 period, even after the short recession that ended in April 2020, jumps are not rare.

3.4 RELATIONSHIPS

Co-movements of the equity market and other factors such as interest rates, inflation, and credit spreads can lead to more severe financial impacts, in addition to heightened volatility and volatility clustering. When assessing the aggregated impact of a risk event, quantification of the diversification benefit is important. A small change in the correlation structure often leads to a significant change in the total required capital. The preferred way to model the relationships in an extreme event is to model them among their underlying risk drivers. Here, a risk driver is to be a random variable that lends itself to univariate statistical modelling and simulation. In this section, relationships among equity returns, 1/10-

year Treasury bond yields, Fed rates, credit spreads, and inflation rates are studied. Contemporary relationships are studied using statistical approaches including correlation matrices and copulas. Temporal relationships are studied using structural models such as the vector autoregressive (VAR) models.

Table 5 lists all the variables used in this section. Given data availability, some variables have shorter periods than others. At a minimum, 32 years of daily data is used which contains several economic cycles to be used as a benchmark to which the post-2020 experience is compared.

Table 5
CORRELATION ANALYSIS HISTORICAL DATA

Variable	Indicator	Notation	Time period	Data source
S&P 500 index return	Large-cap market return	SPX_rtn	Jul 1954 – Apr 2022	FRED Economic Database
Equity index implied volatility	Implied volatility of options on SPX	VIXCLS	Jan 1990 – Apr 2022	CBOE
Fed rate	Federal Funds Effective Rate	DFF	Jul 1954 – Apr 2022	FRED Economic Database
1-year Treasury bond yield	Market Yield on U.S. Treasury Securities at 1-Year Constant Maturity	DGS1	Jan 1962 – Apr 2022	FRED Economic Database
10-year Treasury bond yield	Market Yield on U.S. Treasury Securities at 10-Year Constant Maturity	DGS10	Jan 1962 – Apr 2022	FRED Economic Database
Credit spread	Moody's Seasoned Baa Corporate Bond Yield Relative to Yield on 10-Year Treasury Constant Maturity	BAA10Y	Jul 1986 – Apr 2022	FRED Economic Database
Inflation rate	10-Year Breakeven Inflation Rate	T10YIE	Jan 2003 – Apr 2022	FRED Economic Database

3.4.1 CONTEMPORARY RELATIONSHIP

A correlation matrix contains the correlation coefficients among exposures to individual risk factors with the assumption of linear relationships.

$$\mathrm{RE}_{\mathrm{Total}} = \sqrt{(RE_1 \quad RE_2 \quad RE_3)\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}\begin{pmatrix} RE_1 \\ RE_2 \\ RE_3 \end{pmatrix}}$$

where

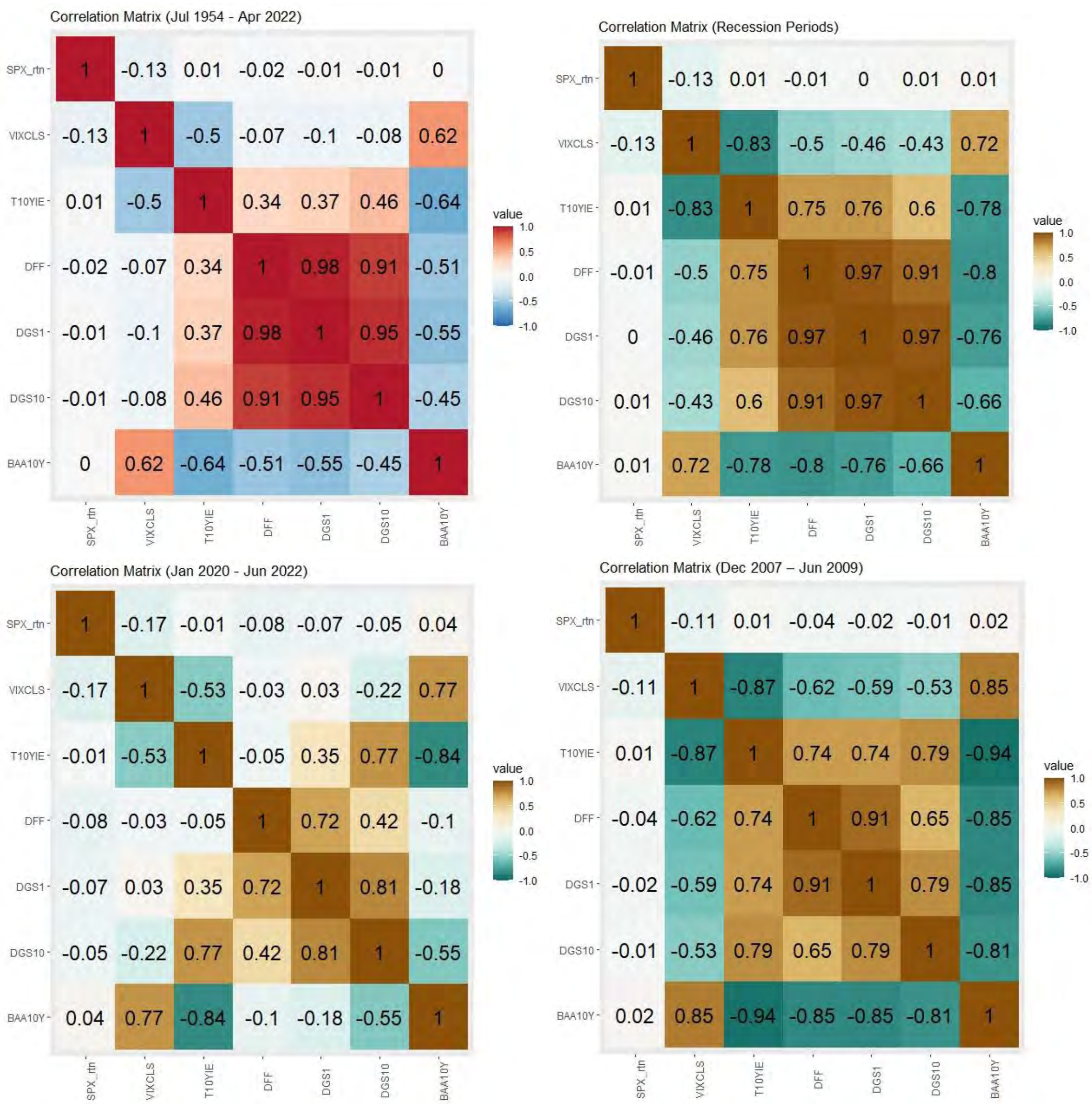
RE_{Total} is the aggregated risk exposure.
RE_i is the risk exposure for risk factor *i*.
ρ_{ij} is the correlation coefficient of risk factors *i* and *j*.

While one correlation matrix can define a set of linear relationships, ideally the correlation matrix at each confidence level is unique to reflect the nonlinear relationship in reality. However, because of the lack of data, it is difficult to construct them credibly. In addition, the correlation between risk drivers is not necessarily the same as the correlation between risk exposures. Therefore, it may need to be adjusted to reflect product features that can strengthen or weaken the relationship.

Figure 11 shows the correlation matrices using all available data, data during recession periods, data of post-2020 period, and data of the 2008 financial crisis. As expected, the correlations observed in recession periods are more significant than using all periods that include both economic recessions and expansions. Because the post-2020 period only has a short recession in early 2020, it is mostly composed of expansion periods. However, the relationship between SPX returns and implied volatility (VIXCLS) is more negative compared to all other study periods. The relationships between VIXCLS and short-term interest rates such as the Fed rate and 1-year Treasury bond yield is weak during the post-2020 period, probably due to the level of interest rate being already low compared to the historical average level, and the equity market could predict the interest rate decisions fairly well in the short term.

Figure 11

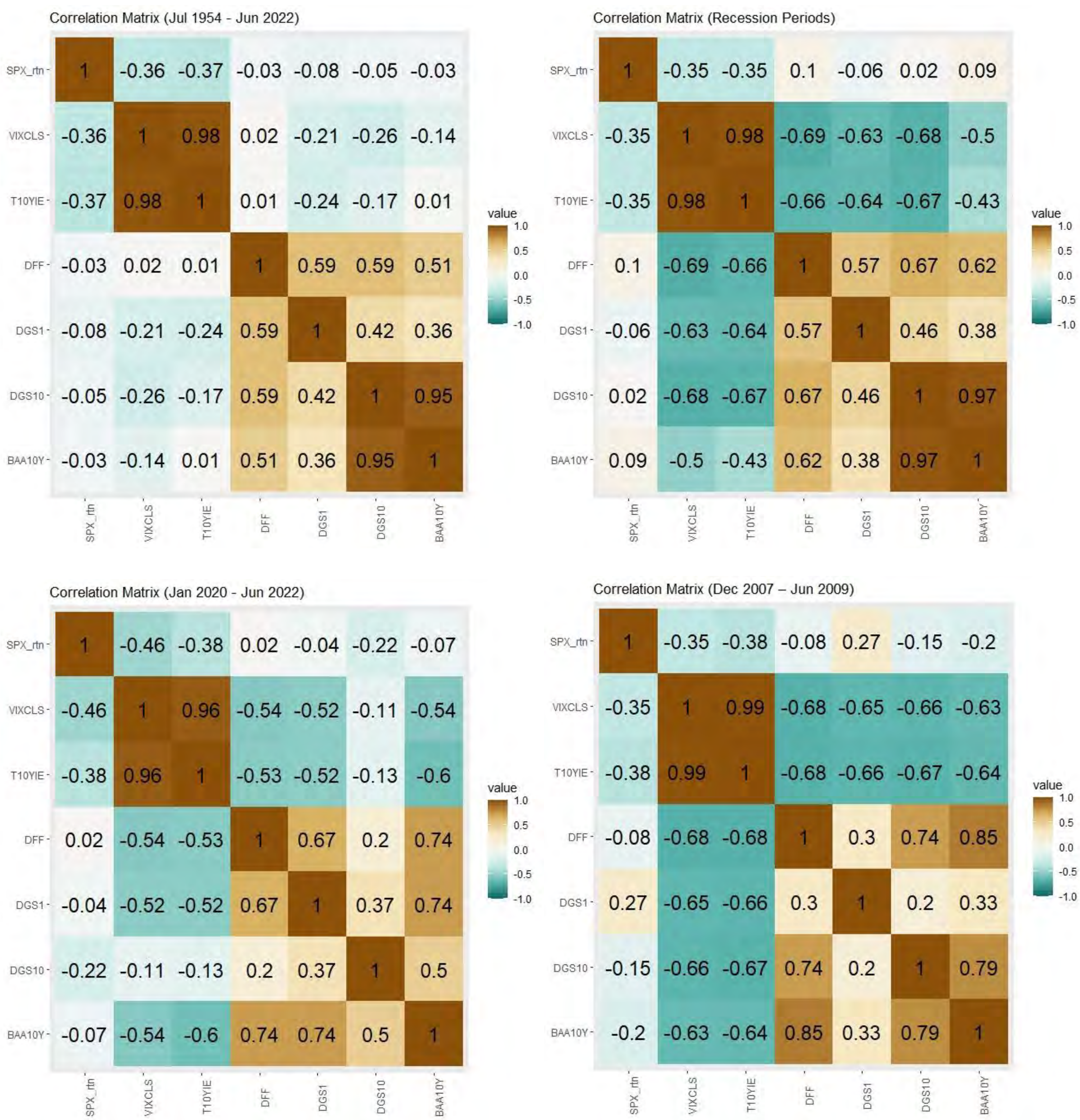
CORRELATION MATRICES USING DAILY DATA



Although an efficient capital market leads to short response time to market movements, sometimes a series of events may lead to lagged responses. Studying monthly data can help reveal the impact of lagged responses. Figure 12 shows the correlation matrices based on monthly data. Similar to the daily data, the relationship between SPX returns and implied volatilities become more negative during the post-2020 period. The relationships between VIXCLS and short-term interest rates such as the Fed rate and 1-year Treasury bond yield is much stronger during the post-2020 period, compared to the observation using the daily data.

Figure 12

CORRELATION MATRICES USING MONTHLY DATA



A single correlation matrix cannot reflect nonlinear relationships. Multiple correlation matrices can be used to describe changing relationships in different situations. However, historical data is usually not sufficient to provide credible estimation of multiple correlation matrices. Alternatively, a copula is used as a general way of formulating a multivariate distribution in such a way that various general types of dependence can be represented. A copula is used to formulate a multivariate distribution via a simple transformation being made of each marginal variable in such a way that each transformed marginal variable has a uniform distribution. Its theoretical foundation is Sklar's theorem of 1959, which says that every multivariate CDF can be written as a function of the marginal distribution functions. For a bivariate CDF, $\mathrm{P}(X \leq x, Y \leq y) = C\big(\mathrm{P}(X \leq x), \mathrm{P}(Y \leq y)\big)$. The copula function C is a parameterized model that describes the relationship of multiple variables. Dependence modelling with copula functions is widely used in applications of financial risk assessment and actuarial analysis.

Table 6 illustrates several copula functions for bivariate analysis, all of which can be extended to multivariate analysis to accommodate three or more variables. With the same marginal distributions, different copulas exhibit different joint distributions. The correlation at the tail implied by the Gumbel copula is the highest in the example. The Clayton copula shows a negative correlation in the example. Although the example is for two variables, copulas can be easily applied to multiple variables as well.

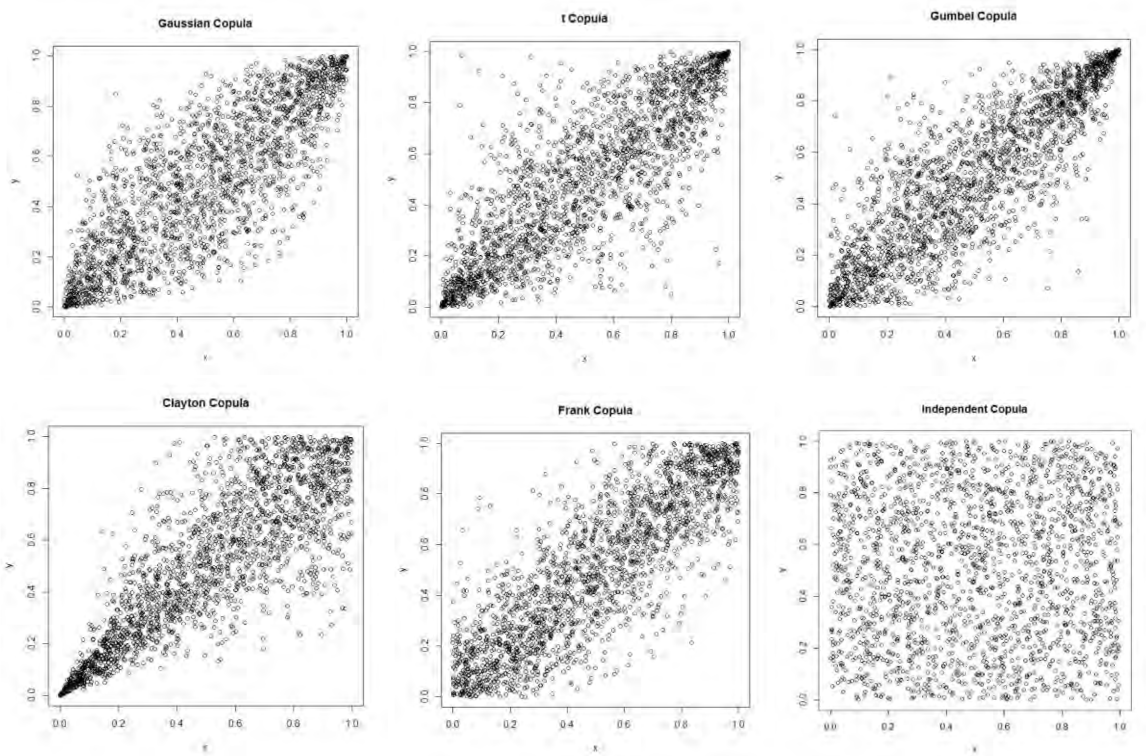
Table 6

COPULA EXAMPLE

Marginal distribution		
$\boldsymbol{u}$	$\mathrm{P}(X \leq x)$	0.95
$\boldsymbol{v}$	$\mathrm{P}(Y \leq y)$	0.95
Joint distribution		
Gaussian copula	Bivariate normal distribution Φ with correlation coefficient ρ $C(u,v) = \Phi_\rho(x,y)$	$\mathrm{P}(X \leq x, Y \leq y) = 0.928$ when $\rho = 0.85$
t copula	Bivariate t distribution with correlation coefficient ρ and the number of the degrees of freedom v	$\mathrm{P}(X \leq x, Y \leq y) = 0.932$ when $\rho = 0.85$ and $v = 5$
Gumbel copula	$C(u,v) = \exp\left(-\left((-\log(u))^\theta + (-\log(v))^\theta\right)^{1/\theta}\right)$	$\mathrm{P}(X \leq x, Y \leq y) = 0.937$ when $\theta = 3$
Clayton copula	$C(u,v) = (u^{-\theta} + v^{-\theta})^{-1/\theta}$	$\mathrm{P}(X \leq x, Y \leq y) = 0.799$ when $\theta = 4$
Frank copula	$C(u,v) = -\frac{1}{\theta}\log\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right)$	$\mathrm{P}(X \leq x, Y \leq y) = 0.916$ when $\theta = 9.5$
Independent	$C(u,v) = u \times v$	$\mathrm{P}(X \leq x, Y \leq y) = 0.9025$

Unlike the correlation matrix approach in which a nonlinear relationship needs to use multiple matrices, a copula can describe a nonlinear relationship. Copulas allow us to parsimoniously reflect a nonlinear dependence in stochastic scenario generations. Figure 13 illustrates a few simulated copulas as used in Table 6. Five sets of simulated data are shown, each set with two variables that have a correlation coefficient of about 0.85, compared to the independent copula with a correlation coefficient of 0. The Gaussian copula models the linear relationship, which is exactly the same as the correlation matrix approach. The t copula has a higher correlation at both ends, the Gumbel copula a higher correlation at the right end, the Clayton copula a higher correlation at the left end, and the Frank copula a lower correlation at both ends. However, like the correlation matrix approach, it is difficult to consider the order and timing of extreme events. Copulas are a complicated statistical concept with many more types and possible applications than discussed above[2].

Figure 13
COPULA SIMULATION



For a comprehensive correlation analysis with many variables of interests, the relationships among those variables are rarely represented by copulas other than Gaussian and *t* copulas. The reason is that the other copulas mentioned above are parsimonious and it is difficult to capture all the variation in the relationships. Alternatively, a few key variables with strong nonlinear relationships may be selected and modelled by copulas. To model the relationships among the seven variables using a single copula, complete data records are used. The goodness-of-fit in terms of copula fitting can be measured by comparing the empirical multivariate distribution and the fitted distribution using statistical tests such as the Cramér–von Mises test and Kolmogorov–Smirnov test. Genest et al. (2009) reviewed and compared a variety of goodness-of-fit tests for copulas. Table 7 lists the calibration results with goodness-of-fit tests.

[2] Roger B. Nelsen's well-known book *An Introduction to Copulas* (1999) provides more theoretical background.

Table 7

COPULA CALIBRATION EXAMPLE

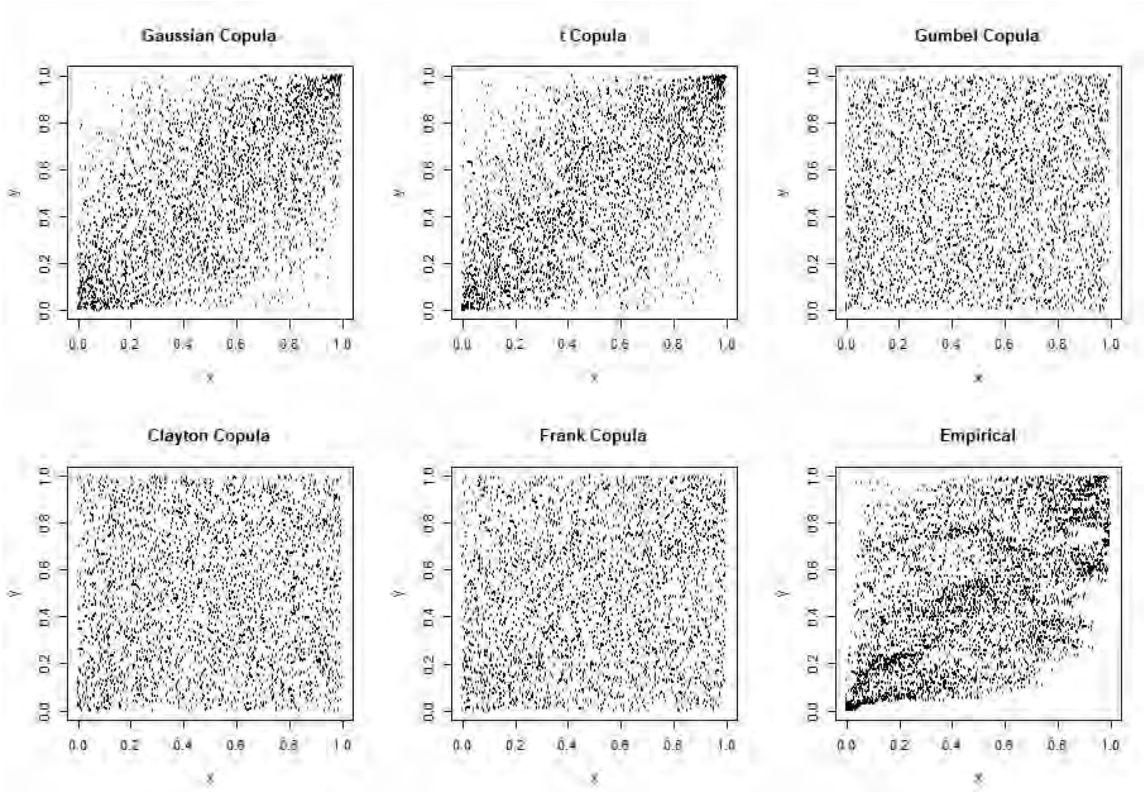
<table>
<tr><th>Copula type</th><th>Parameters</th><th>Sn[1]</th><th>p-value[2]</th></tr>
<tr><td>Gaussian copula</td><td>ρ :
<table>
<tr><th>Variable</th><th>SPX_rtn</th><th>VIXCLS</th><th>T10YIE</th><th>DFF</th><th>DGS1</th><th>DGS10</th><th>BAA10Y</th></tr>
<tr><td>SPX_rtn</td><td>1.00</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>VIXCLS</td><td>-0.10</td><td>1.00</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>T10YIE</td><td>-0.01</td><td>-0.29</td><td>1.00</td><td></td><td></td><td></td><td></td></tr>
<tr><td>DFF</td><td>-0.02</td><td>-0.32</td><td>0.23</td><td>1.00</td><td></td><td></td><td></td></tr>
<tr><td>DGS1</td><td>-0.02</td><td>-0.27</td><td>0.25</td><td>0.91</td><td>1.00</td><td></td><td></td></tr>
<tr><td>DGS10</td><td>-0.02</td><td>-0.20</td><td>0.47</td><td>0.68</td><td>0.72</td><td>1.00</td><td></td></tr>
<tr><td>BAA10Y</td><td>0.02</td><td>0.61</td><td>-0.56</td><td>-0.47</td><td>-0.47</td><td>-0.39</td><td>1.00</td></tr>
</table></td><td>0.0236</td><td>0.868</td></tr>
<tr><td>t copula</td><td>ρ :
<table>
<tr><th>Variable</th><th>SPX_rtn</th><th>VIXCLS</th><th>T10YIE</th><th>DFF</th><th>DGS1</th><th>DGS10</th><th>BAA10Y</th></tr>
<tr><td>SPX_rtn</td><td>1.00</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>VIXCLS</td><td>-0.10</td><td>1.00</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>T10YIE</td><td>-0.01</td><td>-0.25</td><td>1.00</td><td></td><td></td><td></td><td></td></tr>
<tr><td>DFF</td><td>-0.02</td><td>-0.32</td><td>0.24</td><td>1.00</td><td></td><td></td><td></td></tr>
<tr><td>DGS1</td><td>-0.02</td><td>-0.29</td><td>0.27</td><td>0.93</td><td>1.00</td><td></td><td></td></tr>
<tr><td>DGS10</td><td>-0.01</td><td>-0.17</td><td>0.53</td><td>0.64</td><td>0.70</td><td>1.00</td><td></td></tr>
<tr><td>BAA10Y</td><td>0.01</td><td>0.61</td><td>-0.54</td><td>-0.51</td><td>-0.52</td><td>-0.41</td><td>1.00</td></tr>
</table>
$v = 12$</td><td>0.0239</td><td>0.804</td></tr>
<tr><td>Gumbel copula</td><td>$\theta = 1.039$</td><td>0.020</td><td>0.953</td></tr>
<tr><td>Clayton copula</td><td>$\theta = 0.125$</td><td>0.015</td><td>0.943</td></tr>
<tr><td>Frank copula</td><td>$\theta = 1.08$</td><td>0.019</td><td>0.993</td></tr>
</table>

Notes:

1. *Sn: Cramér–von Mises statistic introduced by Genest, Remillard, and Beaudoin (2009).*
2. *P-value: P-value of Sn test using the parameter bootstrapping method introduced by Kojadinovic and Yan (2011).*

To verify whether high correlations observed are captured in any of the calibrated copulas, implied volatilities and credit spreads are used as an illustration of necessary visualization. Figure 14 compares the calibrated copulas and the empirical data. Of all the calibrated copulas, the *t* copula has a higher correlation at both ends, which is consistent with the empirical copula.

Figure 14
COPULA SIMULATION: VIXCLS VS. BAA10Y



Notes:

- *x: VIXCLS*
- *y: BAA10Y*

If a copula is needed to model the nonlinear relationships, the *t* copula seems to be the best choice especially when more than two variables need to be modeled together. Although the copula approach provides more flexibility and preserves the parsimony, it is not an easy task to find the most appropriate copula. The data used for copula calibration may be sparse, and the goodness of fit can be low.

3.4.2 TEMPORAL RELATIONSHIP

Compared to the statistical approaches where contemporary relationships are the main subject to study, structured models are more flexible to deal with both contemporary and temporal relationships at the same time. A monetary policy may be the result of a crisis but can also dampen the impact and duration of the crisis. As evidenced in the recent COVID-19 outbreaks, the proactive monetary policies helped improve market liquidity and reversed the course of a bear market. Correlated simulation models can be used to reflect nonlinear correlation and the timing of events.

A vector autoregressive (VAR) model is used to study nonlinear relationships among equity index return, equity volatility, Fed fund rate, Treasury bond yields, credit spread, and hedgable inflation rate. Although these factors are used in this report, other factors can be included as well, depending on the purpose of a model. A VAR model is used to describe the relationship of the modeled variables based on this historical data. By incorporating lagging variables into the analysis through VAR, relationships among leading, coincident, and lagging variables can be better reflected.

$$\mathbf{E}_t = \mathbf{c} + \sum_{j=1}^{p} A_j \mathbf{E}_{t-j} + \mathbf{e}_t$$

where

$\mathbf{E}_t =$ $(\mathrm{SPX\_rtn}_t, \mathrm{VIXCLS}_t, \mathrm{T10YIE}_t, \mathrm{DEF}_t, \mathrm{DCS1}_t, \mathrm{DCS10}_t, \mathrm{BAA10Y}_t)^T$, a column vector with seven elements as the value of economic factors at time t or during period t;

$\mathbf{c} =$ a column vector with seven elements to represent the constant terms of the seven economic factors;

$A_j =$ a 7×7 matrix containing the model parameters describing the linear dependence of variables with a lag of j; and

$\mathbf{e}_t =$ a column vector with seven elements to store the error terms that cannot be explained by linear models.

The VAR models are fit to two sets of data with the inclusion or exclusion of the post-2020 data. Based on stability tests, a maximum lag (p) of 5 is used. The impact of the post-2020 data is noticeable with an example shown in Table 8 which lists the parameters attached to lagged SPX returns to estimate the implied volatility.

Table 8

VAR(5) SAMPLE MODEL PARAMETERS

Lag	Jan 2002 — Dec 2019	Jan 2002 — Jun 2022
1	0.044	0.544
2	0.049	0.125
3	-0.032	0.161
4	-0.016	-0.151
5	0.032	0.010

Lagged index returns moved from a negative impact to a positive impact on the implied volatility, with the model parameter with a lag of 1 changed from 0.044 to 0.544 in Table 8 as an example. As observed in the post-2020 period, high returns were associated with high volatilities as well. The full tables of model parameters and stability test results can be found in Appendix A.4. Based on the fitted VAR models, the stable values of fundamental risk factors $\bar{\mathbf{E}}$ can be derived. The stable values $\bar{\mathbf{E}}$ can be considered as long-term expected values assuming that the economic system will eventually return to equilibrium.

$$\bar{\mathbf{E}} = \mathbf{c} + \sum_{j=1}^{p} A_j \cdot \bar{E}$$

Table 9 lists the stable values based on VAR(5), along with the historical means, with and without the post-2020 period. The models suggest a higher volatility and slightly lower interest rates based on the conditions at the end of June 2022.

Table 9
VAR(5) STABLE VALUES

Variable	Jan 2002 — Jun 2022		Jan 2002 — Dec 2019	
	Historical Mean	Stable Mean	Historical Mean	Stable Mean
SPX_rtn	9.19%	9.32%	8.74%	8.14%
VIXCLS	24.87%	29.75%	19.25%	19.14%
T10YIE	2.07%	1.75%	2.07%	2.07%
DFF	0.27%	0.41%	1.26%	1.63%
DGS1	0.51%	0.74%	1.40%	1.74%
DGS10	1.42%	1.30%	2.90%	2.94%
BAA10Y	2.28%	2.63%	2.54%	2.50%

Key Findings

1. The relationship between SPX returns and implied volatilities became more negative during the post-2020 period.
2. The daily relationships between VIXCLS and short-term interest rates such as the Fed rate and 1-year Treasury bond yield is weak during the post-2020 period in the short term, probably due to the level of interest rate being already low compared to the historical average level, and the equity market could predict the interest rate decisions fairly well in the short term. The monthly relationships between VIXCLS and short-term interest rates such as the Fed rate and 1-year Treasury bond yield is much stronger during the post-2020 period, compared to the observation using the daily data.
3. Copulas allow us to parsimoniously reflect a nonlinear dependence in stochastic scenario generations. Although the copula approach provides more flexibility and preserves the parsimony, it is not an easy task to find the most appropriate copula. The data used for copula calibration may be sparse, and the goodness-of-fit can be low.
4. A vector autoregressive (VAR) model is used to study nonlinear relationships among equity index return, equity volatility, Fed fund rate, Treasury bond yields, credit spread, and hedgable inflation rate. The impact of the post-2020 data is noticeable. For example, lagged index returns moved to a more positive impact on the implied volatility.

Section 4: Attribution Analysis

In the previous section, we found that the post-2020 period is one of the most extreme periods in terms of market volatility, volatility clustering, and temporal relationships. Although it is safe to say that the pandemic is the trigger of the high volatility, it is helpful to understand the underlying causes and evaluate whether they will have short- or long-term impacts. A data driven approach is used for the attribution analysis. Regression models are used to assess the contribution of a variety of factors to market volatility.

4.1 MARKET DATA

Four categories of data are collected to understand their relationships with the market volatility.

- Economic data that describe the macroeconomic conditions such as economic growths, employment, investment, consumption, debt, and market sentiment.
- Retail investor data that describe the participation of retail investors in trading activities at a high level. Social media data is also used to analyze the impact of retail investors, as explained in Section 4.2.
- Investment style data that track hedge fund assets under management of different investment strategies.
- Event data that indicate extreme events such as pandemics and wars.

Table 10 lists the variables collected and used in the attribution analysis.

Table 10
MARKET DATA DESCRIPTION

Variable	Alias	Type	Periodic change[1]	Frequency[5]	Date Range	Data Source
GPDI	Gross Private Domestic Investment	Economic	Yes	Quarterly	January 1947 – June 2022	FRED Economic Data
T5YIE	5-Year Breakeven Inflation Rate	Economic	No	Daily	January 2003 – June 2022	FRED Economic Data
T10YIE	10-Year Breakeven Inflation Rate	Economic	No	Daily	January 2003 – June 2022	FRED Economic Data
PCE	Personal Consumption Expenditures	Economic	Yes	Monthly	January 1959 – June 2022	FRED Economic Data
CPIAUCSL	Consumer Price Index for All Urban Consumers	Economic	No	Daily	January 1947 – June 2022	FRED Economic Data
W994RC1Q027SBEA	Net lending or net borrowing: Private	Economic	No	Quarterly	January 1960 – June 2022	FRED Economic Data
PSAVERT	Personal Savings Rate	Economic	No	Monthly	January 1959 – June 2022	FRED Economic Data
MTSDS133FMS	Federal Surplus or Deficit	Economic	No	Monthly	October 1980 – June 2022	FRED Economic Data
DEF	Federal Funds Effective Rate	Economic	No	Daily	July 1954 – June 2022	FRED Economic Data
DSPI	Disposable Personal Income	Economic	Yes	Monthly	January 1959 – June 2022	FRED Economic Data
UMCSENT	University of Michigan: Consumer Sentiment	Economic	Yes	Monthly	November 1952 – June 2022	FRED Economic Data
GFDEGDQ188S	Federal Debt: Total Public Debt as Percent of Gross Domestic Product	Economic	No	Quarterly	January 1966 – June 2022	FRED Economic Data
A191RL1Q225SBEA	Real Gross Domestic Product, Percent Change from Preceding Period	Economic	No	Quarterly	April 1947 – June 2022	FRED Economic Data
covid_case_us	U.S. daily COVID cases	Event	No	Daily	January 2020 – June 2022	JHU CSSE COVID-19 Data

covid_death_us	U.S. daily COVID deaths	Event	No	Daily	January 2020 – June 2022	JHU CSSE COVID-19 Data
Retail_Share		Retail	No	Yearly	January 2013 – June 2022	Bloomberg Intelligence
RH_MAU	Monthly active users of Robinhood	Retail	No	Quarterly	October 2015 – June 2022	Statista and Robinhood Earnings Report
RH_AUC	Assets under custody of Robinhood	Retail	No	Quarterly	January 2013 – June 2022	Statista and Robinhood Earnings Report
gt_ukraine	Google trend index: Ukraine	Event	No	Daily	January 2004 – June 2022	Google Trends
gt_pandemic	Google trend index: Pandemic	Event	No	Daily	January 2004 – June 2022	Google Trends
gt_covid	Google trend index: COVID	Event	No	Daily	January 2004 – June 2022	Google Trends
gt_market_crash	Google trend index: market crash	Retail	No	Daily	January 2004 – June 2022	Google Trends
gt_inflation	Google trend index: inflation	Economic	No	Daily	January 2004 – June 2022	Google Trends
gt_job	Google trend index: job	Economic	No	Daily	January 2004 – June 2022	Google Trends
gt_interest_rate	Google trend index: interest rate	Economic	No	Daily	January 2004 – June 2022	Google Trends
gt_stock_market	Google trend index: stock market	Retail	No	Daily	January 2004 – June 2022	Google Trends
ma_debit	Debit Balances in Customers' Securities Margin Accounts	Economic	Yes	Monthly	January 1997 – August 2022	FINRA margin statistics
ma_credit	Free Credit Balances in Customers' Securities Margin Accounts	Economic	Yes	Monthly	January 1997 – August 2022	FINRA margin statistics
AUM_HF[2]	Asset under management: Hedge Funds	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_fof	Asset under management: Fund of Funds	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_BSB	Asset under management: Balanced	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge

	(Stock and Bonds)					
AUM_Con	Asset under management: Convertible Arbitrage	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_Dis	Asset under management: Distressed Securities	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_EM	Asset under management: Emerging Markets	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_EMA	Asset under management: Emerging Markets - Asia	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_EMLA	Asset under management: Emerging Markets – Latin America	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_EMG	Asset under management: Emerging Markets – Global	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_EMEE	Asset under management: Emerging Markets – Eastern Europe	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_ELO	Asset under management: Equity Long Only	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_ELB	Asset under management: Equity Long Bias	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_ELS	Asset under management: Equity Long/Short	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_EMN	Asset under management: Equity Market Neutral	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_ED	Asset under management: Event Driven	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_FI	Asset under management: Fixed Income	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_MA	Asset under management:	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge

	Merger Arbitrage					
AUM_Mac	Asset under management: Macro	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_MS	Asset under management: Multi-Strategy	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_OS	Asset under management: Options Strategies	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_Other[3]	Asset under management: Other	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge
AUM_SS[4]	Asset under management: Sector Specific	Investment Style	Yes	Quarterly	January 2000 – June 2022	BarclayHedge

Notes:

1. *Some variables have been transformed to relative changes from the beginning to the end of each period and indicated as "Yes". Some variables are already represented as relative changes in their original form and are not indicated as "Yes."*
2. *AUM_HF excludes fund of funds assets.*
3. *AUM_Other includes funds categorized as Algorithmic, Closed-end funds, Dividend Capture, Equity Dedicated Short, Equity Short-Bias, Mutual Funds/ETFs, No Category, PIPEs (Regulation D), Replication, and Tail Risk.*
4. *AUM_SS includes sector funds categorized as Energy, Environment, ESG, Farming, Financial, Health Care/Biotech, Metals/Mining, Miscellaneous, Natural Resources, Real Estate, and Technology.*

Before using these variables for attribution analysis, it is helpful to understand their correlations with the market volatility in the past. Table 11 lists the variables that showed high correlation with VIXCLS, either positive or negative. Both the concurrent correlations (same day) and lagged correlations are calculated to evaluate the potential predicting power of these explanatory variables. For example, the correlation coefficient between explanatory variable T5YIE at month *m*-1 and the target variable market volatility at month *m* is ***-0.55***, as listed in the following table.

Table 11

MARKET DATA CROSS CORRELATION

1-month lag[1]		2-week Lag		1-week lag		3-day lag		1-day lag		same day	
Variable	Correl[2]	Variable	Correl	Variable	Correl	Variable	Correl	Variable	Correl	Variable	Correl
T5YIE	***-0.55***	T5YIE	-0.57	T5YIE	-0.58	T5YIE	-0.58	AUM_ELB	-0.59	AUM_ELB	-0.60
T10YIE	-0.53	T10YIE	-0.55	AUM_ELB	-0.57	AUM_ELB	-0.58	T5YIE	-0.58	T5YIE	-0.58
AUM_ELB	-0.44	AUM_ELB	-0.53	T10YIE	-0.55	T10YIE	-0.55	T10YIE	-0.55	T10YIE	-0.55
AUM_ELS	-0.37	AUM_ELS	-0.45	AUM_ELS	-0.47	AUM_ELS	-0.48	AUM_ELS	-0.49	AUM_ELS	-0.50
RH_AUC	-0.37	AUM_HF	-0.40	AUM_HF	-0.44	AUM_HF	-0.45	AUM_HF	-0.47	AUM_HF	-0.47
AUM_ED	-0.34	AUM_ED	-0.38	AUM_ED	-0.40	AUM_ED	-0.41	AUM_SS	-0.42	AUM_SS	-0.43
AUM_fof	-0.33	AUM_fof	-0.37	AUM_SS	-0.39	AUM_SS	-0.41	AUM_ED	-0.42	AUM_ED	-0.42
AUM_HF	-0.32	AUM_SS	-0.35	AUM_fof	-0.39	AUM_fof	-0.40	AUM_fof	-0.40	ma_debit	-0.41
AUM_EMN	-0.32	AUM_EMN	-0.35	AUM_EMN	-0.36	ma_debit	-0.38	ma_debit	-0.40	AUM_fof	-0.41
Contraction	0.34	RH_MAU	0.40	RH_MAU	0.40	RH_MAU	0.41	RH_MAU	0.41	RH_MAU	0.41
RH_MAU	0.39	gt_market_crash	0.40	gt_market_crash	0.45	gt_market_crash	0.46	gt_pandemic	0.45	gt_pandemic	0.44
gt_stock_market	0.45	gt_pandemic	0.48	gt_pandemic	0.47	gt_pandemic	0.46	gt_market_crash	0.46	gt_market_crash	0.46
gt_pandemic	0.46	Retail_Share	0.49	Retail_Share	0.49	Retail_Share	0.48	Retail_Share	0.48	Retail_Share	0.48
Retail_Share	0.49	gt_stock_market	0.60	gt_stock_market	0.63	gt_stock_market	0.63	gt_stock_market	0.63	gt_stock_market	0.63
BAA10Y	0.64	BAA10Y	0.66	BAA10Y	0.65	BAA10Y	0.65	BAA10Y	0.64	BAA10Y	0.64

Notes:

1. *The lag is the lag between the explanatory variables and the target variable VIXCLS. Explanatory variables precede the target variable to evaluate potential predicting power.*
2. *Correlation coefficient*
3. *Daily data from January 2000 to June 2022 is used.*

For economic variables, 5-/10-year breakeven inflation rates are negatively correlated with the market volatility and 10-year BAA-rated corporate bond credit spreads are positively correlated with the market volatilities. For retail investor variables, retail share of trading and Robinhood monthly active users are positively related to market volatility which indicates that increasing retail investors may contribute to market volatility. This is also consistent with the positive correlation between Google Trend statistics (stock market and market crash) and the market volatility. Pandemic events based on Google Trend statistics are also positively correlated with market volatility. The AUM of certain hedge fund investment styles also showed negative correlation with the market volatility. Reduction in AUM may indicate the presence of bear market and heightened market volatility.

Similar analysis is performed on data from January 2020 to June 2022 to understand if the driving factors changed from before. Table 12 shows the variables with noticeable correlation with the market volatility, both concurrent and temporal.

Table 12

MARKET DATA CROSS CORRELATION: PANDEMIC PERIOD

1-month lag[1]		2-week Lag		1-week lag		3-day lag		1-day lag		same day	
Variable	Correl[2]	Variable	Correl	Variable	Correl	Variable	Correl	Variable	Correl	Variable	Correl
AUM_ELB	-0.41	AUM_ELB	-0.48	AUM_ELB	-0.52	AUM_ELB	-0.53	AUM_ELB	-0.59	AUM_ELB	-0.54
GPDI	-0.40	AUM_ELS	-0.41	AUM_ELS	-0.43	AUM_ELS	-0.44	AUM_ELS	-0.58	AUM_ELS	-0.45
AUM_ELS	-0.34	GPDI	-0.39	AUM_HF	-0.40	AUM_HF	-0.41	AUM_HF	-0.55	AUM_HF	-0.43
A191RL1Q225SBEA	-0.33	AUM_HF	-0.37	GPDI	-0.38	ma_debit	-0.38	ma_debit	-0.49	ma_debit	-0.41
AUM_ED	-0.30	AUM_ED	-0.34	AUM_ED	-0.36	GPDI	-0.38	AUM_SS	-0.47	AUM_SS	-0.39
AUM_fof	-0.30	AUM_fof	-0.34	ma_debit	-0.36	AUM_SS	-0.37	GPDI	-0.42	AUM_ED	-0.38
AUM_HF	-0.29	AUM_EMN	-0.32	AUM_fof	-0.36	AUM_ED	-0.37	AUM_ED	-0.42	GPDI	-0.37
AUM_EMN	-0.29	AUM_SS	-0.32	AUM_SS	-0.36	AUM_fof	-0.36	AUM_fof	-0.40	AUM_fof	-0.37
AUM_MA	-0.27	A191RL1Q225SBEA	-0.31	AUM_EMN	-0.33	AUM_EM	-0.34	AUM_EM	-0.40	AUM_EM	-0.36
AUM_SS	-0.24	AUM_EMA	-0.29	AUM_EM	-0.32	AUM_EMN	-0.34	AUM_EMA	-0.39	AUM_EMA	-0.34
MTSDS133FMS	-0.23	AUM_EM	-0.29	AUM_EMA	-0.32	AUM_EMA	-0.33	AUM_EMN	-0.37	AUM_EMN	-0.34
CPIAUCSL	-0.22	ma_debit	-0.29	A191RL1Q225SBEA	-0.30	AUM_EMLA	-0.30	AUM_EMLA	-0.37	AUM_EMLA	-0.32
AUM_EMA	-0.22	AUM_MA	-0.28	AUM_EMLA	-0.28	AUM_EMEE	-0.30	AUM_EMEE	-0.35	AUM_EMEE	-0.32
Contraction	0.46	Contraction	0.49	Contraction	0.50	Contraction	0.50	Contraction	0.51	Contraction	0.51
BAA10Y	0.71	BAA10Y	0.73	BAA10Y	0.73	BAA10Y	0.72	BAA10Y	0.71	BAA10Y	0.71

Notes:

1. *The lag is the lag between the explanatory variables and the target variable VIXCLS. Explanatory variables precede the target variable to evaluate potential predicting power.*
2. *Correlation coefficient*
3. *Daily data from January 2020 to June 2022 is used.*

It is important to understand that the generally heightened volatility level during the pandemic is not expected to be fully explained by the variables in Table 12. For example, the retail investors' higher participation existed during the entire period with little variation and therefore having lower correlation. Different from the variables in Table 11, economic variables such as contraction has a higher correlation with market volatility. Private investment (GPDI), real GDP growth rate (A191RL1Q225SBEA), and Federal deficit (MTSDS133FMS) had higher negative correlation during the recent pandemic. The leverage level of investors measured by debit balances in securities margin accounts (ma_debit) is also a leading indicator of market volatility.

Key Findings

1. Four categories of data are collected to understand their relationships with the market volatility.
    - Economic data that describe the macroeconomic conditions.
    - Retail investor data that describe the participation of retail investors in trading activities at a high level.
    - Investment style data that tracks hedge fund assets under management of different investment strategies.
    - Event data that indicates extreme events such as pandemics and wars.

2. Both the concurrent correlations (same day) and lagged correlations are used to evaluate the potential predicting power of these explanatory variables. Some correlations in the post-2020 period differ from before. Contraction has a higher correlation with market volatility. Private investment (GPDI), real GDP growth rate and Federal deficit had higher negative correlation during the recent pandemic. The leverage level of investors measured by debit balances in securities margin accounts is a leading indicator of market volatility.

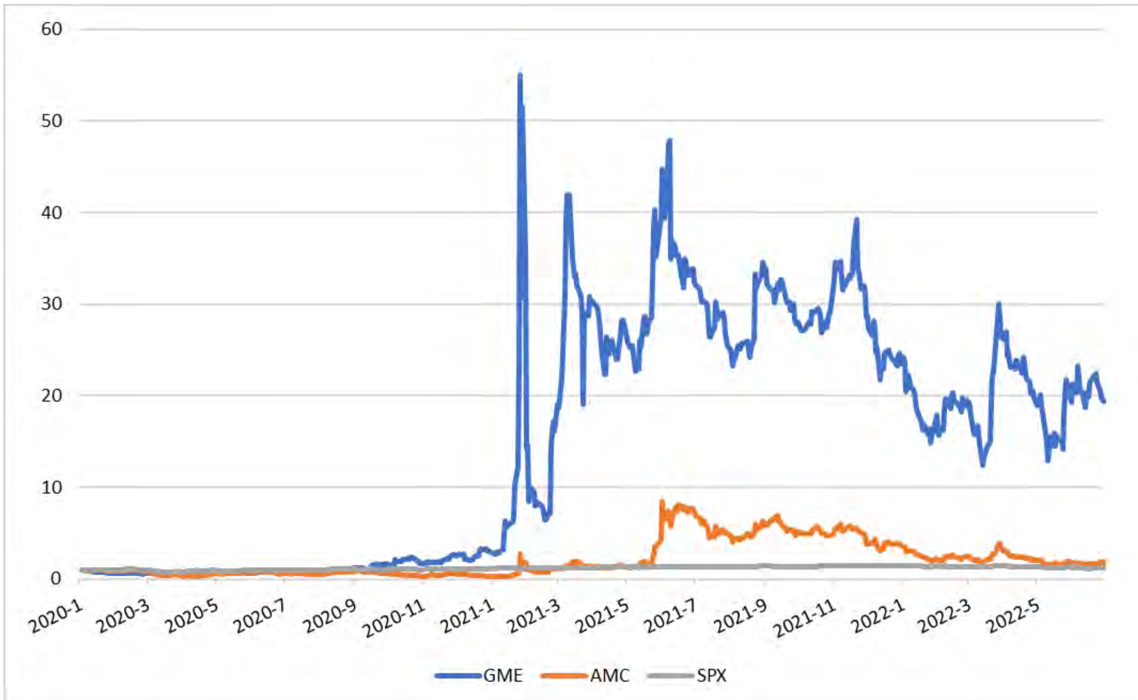
4.2 REDDIT DATA

During the pandemic, retail investors had an increasing influence on the public equity market, with retail investors' share of trading volume climbing by 10% to 25%. People have more time or opportunities to monitor the market while working from home. Tech companies such as Robinhood and WealthSimple made trading convenient and with a low commission. Social media made it easier to retrieve and share information. With less public life, people without financial distress may also have a higher risk appetite to participate in activities such as stock trading.

There is no doubt that retail investors can cause extremely high volatility in single-stock trading. During the pandemic, meme stocks such as GME and AMC experienced much higher volatility than the general market. Figure 15 shows the daily prices of GME, AMC and SPX, which all of them normalized to 1 at the beginning of the study period. While SPX has an annualized daily volatility of 25.5%, GME has a volatility of 194.0% and AMC has a volatility of 246.8%.

Figure 15

DAILY PRICE HISTORY OF GME, AMC AND SPX (JANUARY 2020 TO JUNE 2022)

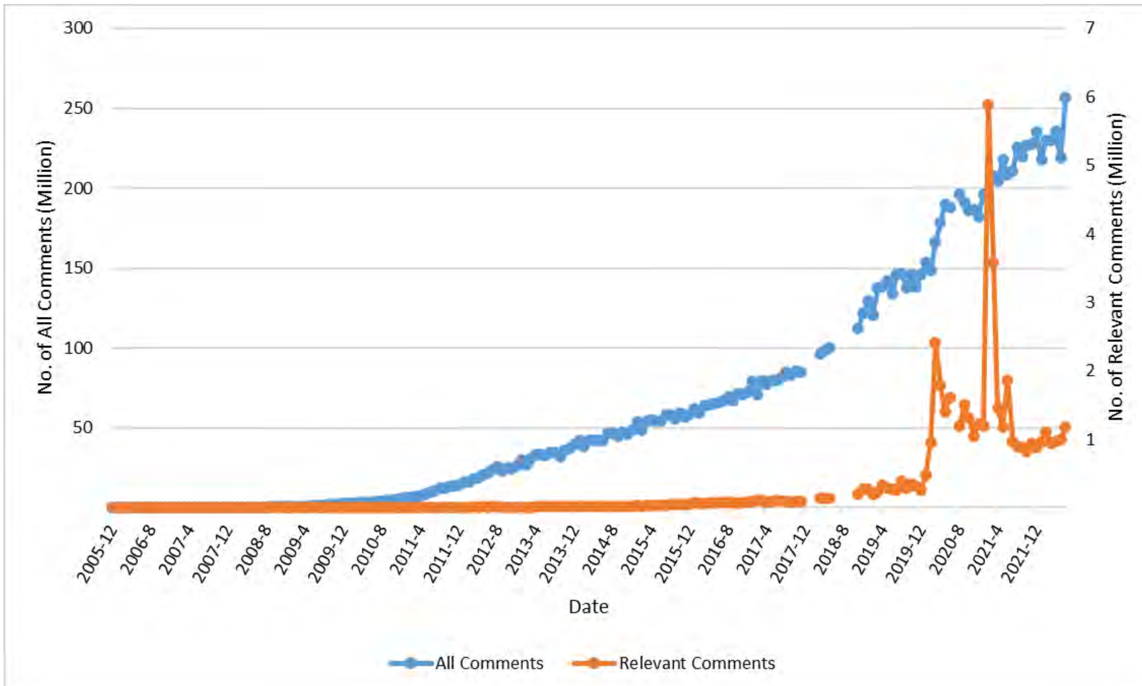


Note: prices have been normalized to 1 at the beginning of 2020.

Retail investors came up with complicated investment strategies to take advantage of the large short positions on the meme stocks held by hedge funds. By purchasing the stocks in a coordinated way and out-of-money call options, the stock price enters into an upward spiral with the short position holders buying the stocks as well to offset their short positions. Although the market cap of these two meme stocks is not material compared to the entire public equity market (around 0.15% of S&P 500 market cap at its highest share), it may indicate the increasing impact of retail investors less significantly for individual stocks but covering more stocks.

To assess the impact of the active participation of retail investors on the general market volatility, Reddit data from December 2005 to June 2022 is used. Reddit is a network of communities where people have common interests and share information. Figure 16 shows the increasing amount of all available Reddit data, with a continuous upward trend.

Figure 16
MONTHLY REDDIT COMMENTS (DECEMBER 2005 TO JUNE 2022)



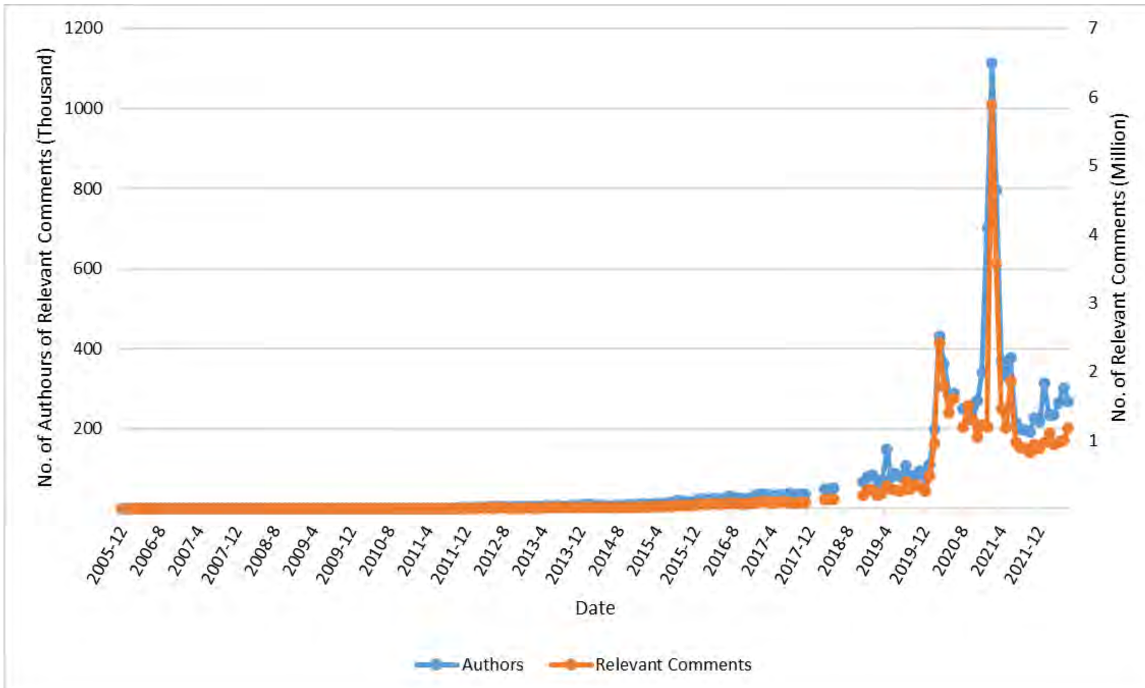
In addition to the total number of comments in the Reddit dataset, the number of relevant comments is also shown in Figure 16. Reddit has a few big communities focusing on investing, such as wallstreetbets, investing, ETFs, and StockMarket. It is also the place where retail investors coordinated to discuss the strategies for the meme stocks. Comments of these four subreddits are considered relevant and used to analyze the relationship between Reddit comments and market volatility. The largest spike in February 2021 happened concurrently with the price spikes of the meme stocks shown in Figure 15.

The comments are cleaned and summarized to be used together with the market data discussed in Section 4.1. On a daily basis, the following explanatory variables are constructed:

- Number of authors that published a comment.
- Number of comments.
- Average score of comments. Each comment is assigned a score by Reddit based on the difference between the number of upvotes and downvotes. It is an indication of the comment's popularity.
- Average sentiment of comments. Each comment is analyzed and assigned a sentiment score. The scores are then averaged each day.
- Weighted average sentiment of comments. The sentiment score is averaged with the Reddit score as the weight.
- Frequency of key words. Word frequency is computed, and the daily frequency of the top frequent words are used to examine potential impact on market volatility.

Figure 17

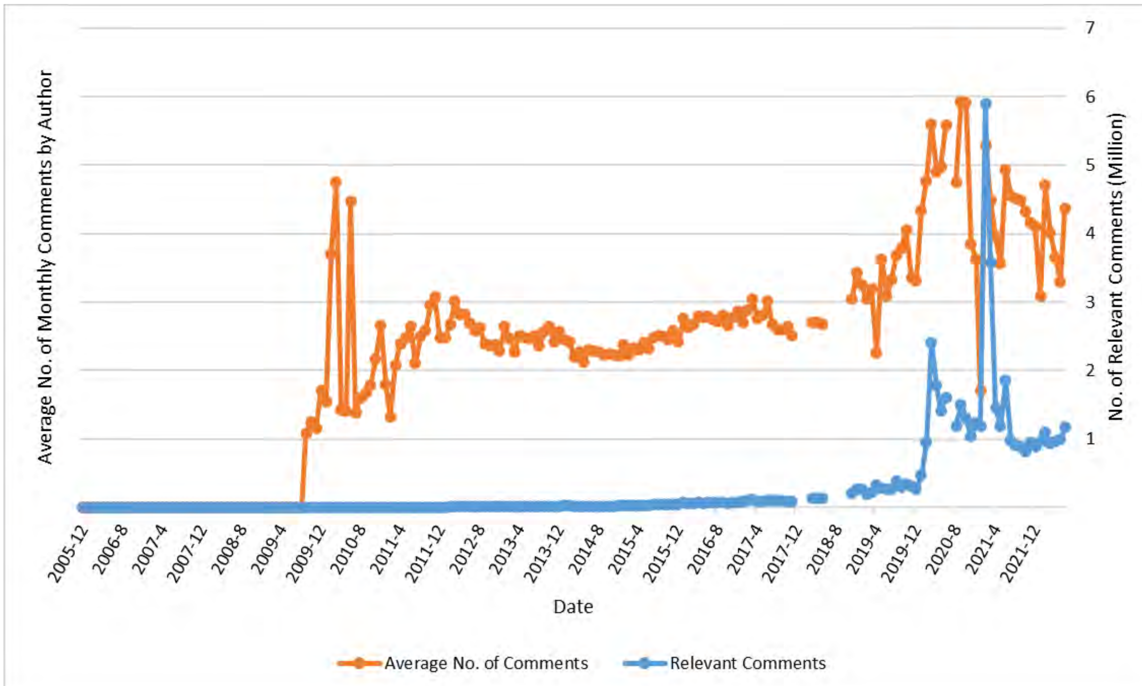
MONTHLY REDDIT COMMENTS AND AUTHORS (DECEMBER 2005 TO JUNE 2022)



Figue 17 shows that the number of authors and the number of comments are highly correlated. It means that the number of comments are largely caused by the participation of more people. This is further confirmed in Figure 18 which shows that the average number of comments per author is not highly correlated with the number of relevant comments.

Figure 18

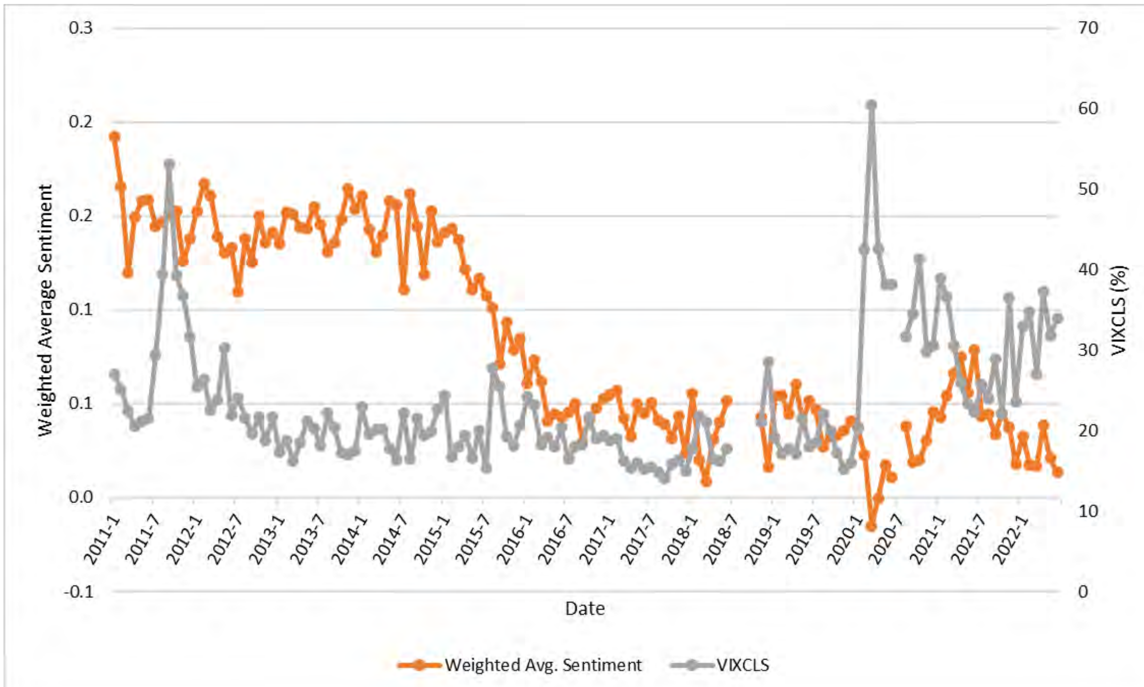
MONTHLY REDDIT AUTHORS AND AVERAGE COMMENTS PER AUTHOR (DECEMBER 2005 TO JUNE 2022)



The sentiment of relevant comments is also helpful for understanding their relationshp with market volatility. Figure 19 shows the weighted average monthly sentiment and VIXCLS, the 30-day expected market volatiilty of the U.S. stock market. A much stronger negative correlation is observed during the recent pandemic.

Figure 19

REDDIT COMMENT SENTIMENT VS. VIXCLS (JANUARY 2011 TO JUNE 2022)



Notes:

1. *Weighted average sentiment uses Reddit score as the weight to calculate average sentiment of relevant comments.*
2. *Relevant comments before 2011 have a maximum monthly count of 200 and are excluded from this analysis.*

To assess any potential predictive power of Reddit comments and any structural change during recent pandemics, cross correlations are examined in Table 13.

Table 13

CROSS CORRELATION BETWEEN REDDIT COMMENT SUMMARY VARIABLES AND VIXCLS

Variable	1-month lag[1]		2-week Lag		1-week lag		1-day lag		same day	
	Pandemic[4]	All[5]	Pandemic	All	Pandemic	All	Pandemic	All	Pandemic	All
No. of comments[2]	0.02	0.27	0.05	0.29	0.09	0.31	0.13	0.33	0.15	0.34
No. of authors	0.02	0.30	0.03	0.31	0.06	0.33	0.10	0.34	0.12	0.35
Sentiment[3]	-0.22	-0.07	-0.42	-0.11	-0.49	-0.12	-0.51	-0.14	-0.52	-0.14

Notes:

1. *The lag is the lag between the explanatory variables and the target variable VIXCLS. Explanatory variables precede the target variable to evaluate potential predicting power.*
2. *No. of relevant comments.*
3. *Weighted average sentiment using Reddit scores as the weight.*
4. *Pandemic period goes from January 2020 to June 2022.*
5. *All period goes from January 2011 to June 2022.*
6. *Daily data is used.*

It is interesting to know that the cross correlations changed significantly during the recent pandemic, with movements in both magnitude and direction. The number of relevant comments and authors are less correlated with the market volatilities. It may indicate that people are participating in the investment discussion not solely due to extreme events but a long-term commitment to the investment community.

The sentiment of comments has a more negative correlation with the market volatility during the pandemic, even with a 2-week lag. This may indicate the predicting power of the sentiment and the volatility clustering.

As part of the analysis, word frequency is also used to evaluate the relationships between Reddit comments and market volatility. Out of 51,888,360 relevant comments, the top 200 words are identified and shown in Figure 20. Frequent words such as “buy”, “sell”, “call”, “put”, “bull”, and “bear” can be meaningful indicators of market volatility.

Figure 20

RELEVANT REDDIT COMMENT WORD CLOUD

With an overview of explanatory variables from the Reddit data, the cross correlations between explanatory variables and the market volatility can be investigated to identify any strong relationships, as shown in Table 14.

Table 14

CROSS CORRELATION BETWEEN KEY WORDS AND VIXCLS (TOP 10 DURING RECENT PANDEMIC)

1-month lag[1]		2-week Lag		1-week lag		3-day lag		1-day lag		same day	
Word	Correl[2]	Word	Correl	Word	Correl	Word	Correl	Word	Correl	Word	Correl
oil	0.42	fed	0.51	put	0.63	put	0.71	put	0.78	put	0.82
futur	0.36	bear	0.51	spi	0.63	spi	0.69	spi	0.75	spi	0.79
bear	0.34	futur	0.49	fed	0.61	futur	0.65	futur	0.70	bottom	0.73
call	0.29	spi	0.48	futur	0.60	fed	0.62	bottom	0.68	futur	0.73
fed	0.28	put	0.46	bear	0.58	home	0.61	bull	0.68	bull	0.71
home	0.27	home	0.46	home	0.57	bear	0.61	bear	0.67	fed	0.68
spi	0.25	bull	0.43	bull	0.54	bottom	0.60	fed	0.65	bear	0.68
bull	0.23	recess	0.40	bottom	0.52	bull	0.59	home	0.64	home	0.67
red	0.22	oil	0.38	recess	0.52	recess	0.57	recess	0.63	recess	0.67
dump	0.20	pump	0.37	pump	0.48	pump	0.55	pump	0.61	pump	0.65

Notes:

1. *The lag is the lag between the explanatory variables (word frequency) and the target variable VIXCLS. Explanatory variables precede the target variable to evaluate potential predicting power.*
2. *Correlation coefficient.*
3. *Daily data from January 2020 to June 2022 is used.*

Details of natural language processing (NLP) methods applied can be found in Appendix B. Reddit Data Analysis.

Key Findings

1. There is no doubt that retail investors can cause extremely high volatility in single-stock trading. During the pandemic, meme stocks such as GME and AMC experienced much higher volatility than the general market. Although the market cap of these two meme stocks is not material compared to the entire public equity market (around 0.15% of S&P 500 market cap at its highest share), it may indicate the increasing impact of retail investors less significantly for individual stocks but covering more stocks.
2. Reddit has a few big communities focusing on investing, such as wallstreetbets, investing, ETFs, and StockMarket. Comments of these four subreddits are considered relevant and used to analyze the relationship between Reddit comments and market volatility. The largest spike of relevant comments in February 2021 happened concurrently with the price spikes of the meme stocks.
3. The sentiment of relevant comments is also helpful for understanding their relationship with market volatility. A much stronger negative correlation between weighted average monthly sentiment and VIXCLS, the 30-day expected market volatility of the U.S. stock market, is observed during the recent pandemic.

4.3 PREDICTIVE MODELING

With the collected data, predictive models are used to understand the contribution of each explanatory variable to the market volatility. It is important to know that the attribution analysis is not necessarily a cause-and-effect analysis. In general, a cause-and-effect relationship needs to be proven by showing that the cause always leads to the effect and happens before the effect, and there are no other factors that can explain the effect. It usually requires a control group and a treatment group that are identical except the cause that we want to prove. This is possible in scientific research such as vaccine effectiveness but is extremely difficult to apply to economic analysis because only one path of reality can be observed.

<u>Modeling Choices</u>

However, while it is unlikely to fully prove the cause-and-effect relationships, analyzing the relationships between explanatory variables and the market volatility can be insightful. To make the analysis as useful as possible, two modeling choices have been made:

- Daily frequency is used to make sure temporal relationships can be evaluated. Cause-and-effect relationships are usually temporal relationships even though the reaction time between cause and effect can be little. Daily frequency is highest frequency that data is available. If a lower frequency such as monthly frequency is used, with events happening in the same month, it is difficult to conclude which one is the cause and which one is the effect based purely on the data analysis. For example, a material monthly loss of the capital market may cause the central bank to reduce the interest rates. Alternatively, reducing the interest rates may be seen as an indication of weak economy and causes market downturn.
- Instead of analyzing the contemporary relationships between explanatory variables and market volatilities, explanatory variables during previous periods are used to explain current market volatility. This temporal precedence ensures that the cause and effect are not swapped. 1-3-day, 1-2-week, and 1-month lags are used for each explanatory variables including both market data in Section 4.1 and Reddit data in Section 4.2.

<u>Data Processing</u>

Before using the data for attribution analysis, data is processed to make it more suitable for regression analysis.

- Missing data treatment. Missing data is quite common, with historical data of some explanatory variables compiled later than others. To strike a balance, data records before 2000 are removed from the predictive modeling exercise, given many variables, especially retail investor and investment style data not being available. For missing values after 1999, it can be caused by lost or bad data records, and data frequency lower than the daily frequency. They cannot be simply removed because autocorrelation and cross correlation are the key patterns to recognize in this analysis. Interpolation is used to estimate and replace these missing values. Cubic spline interpolation is used to be able to capture the speed of change, compared to linear interpolation where a constant speed of change is assumed. This also adds some “randomness” in the dataset which is beneficial for overcoming the issue of overfitting.

- Data normalization. When explanatory variables have different levels of magnitude, they may need to be normalized so that the parameter calibration will not be dominated by a small portion of the variables, and therefore better reflect the relationship between response variable and explanatory variables. Standardized scaling is used as defined below. It is a commonly used method and a reasonable choice for cases with and without outliers.

$$X' = \frac{X - \mu_x}{\sigma_x}$$

Where
μ_x: mean of X variable
σ_x: standard deviation of X variable

- Data split. The entire dataset is split into two subsets: training dataset and validation dataset. The splitting is performed by data record, with 80% of the records (in-the-sample data) in the training dataset and 20% of the records (out-of-sample data) in the validation dataset. Only the training dataset is observable during the model training process, with the validation dataset used to evaluate the accuracy of calibrated models.

Model Types

Given the nonlinear relationships observed during data exploration in previous sections, in addition to linear models, other model types are used for attribution analysis as well, as introduced below.

Linear regression is the simplest yet most powerful parametric model. It assumes a linear relationship between explanatory variables and response variable.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n$$

Model parameters can be estimated by minimizing the squared errors. The simple linear regression also has many variants including Lasso, Ridge regression and Elastic Net with different methods of regularization to prevent overfitting. By adding a penalty for model complexity into the error function, regularization can be used in many predictive models to mitigate the risk of overfitting. For example, ridge regression is a version of linear regression with regularization. Normal regularization includes L1 regularization, which uses the sum of the absolute value of parameters, as in the LASSO model, and L2 regularization, which uses the sum of the squared value of parameters, as in ridge regression. Elastic Net models use both.

Linear Regression: $\min_{\beta} \sum_{j=1}^{m} \left(Y_j - \sum_{i=1}^{n} x_i^j \beta_i\right)^2$

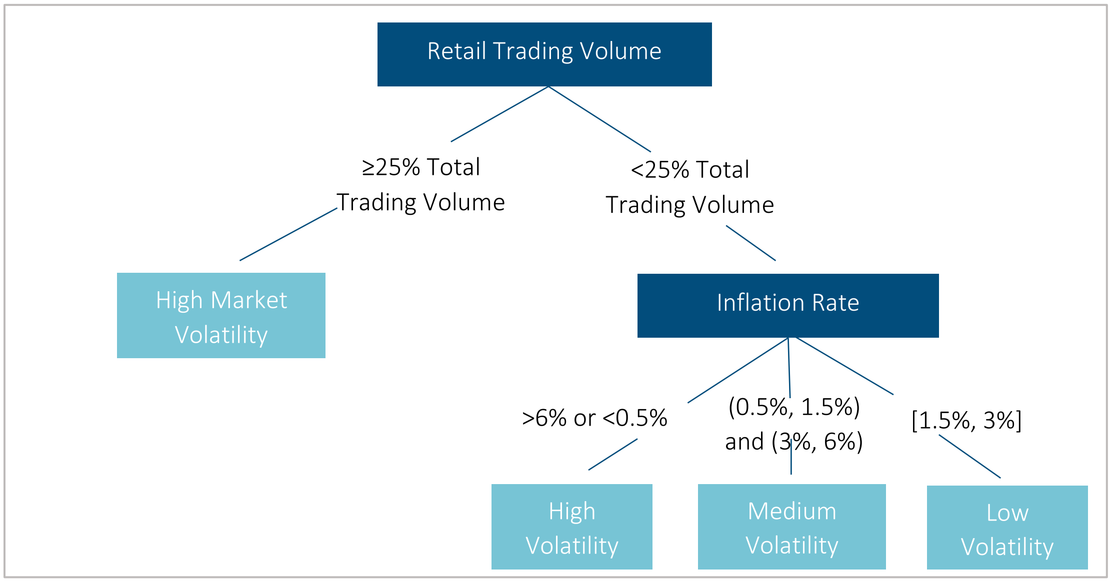
Lasso Regression: $\min_{\beta} \sum_{j=1}^{m} \left(Y_j - \sum_{i=1}^{n} x_i^j \beta_i\right)^2 + \lambda \sum_{i=1}^{n} |\beta_i|$

Ridge Regression: $\min_{\beta} \sum_{j=1}^{m} \left(Y_j - \sum_{i=1}^{n} x_i^j \beta_i\right)^2 + \lambda \sum_{i=1}^{n} \beta_i^2$

Parameter λ controls the weight of the penalty.

Unlike linear models, **tree-based models** switch from formulas to decision rules for prediction. In a tree, leaves represent different subgroups and branches represent the rules to split into subgroups based on explanatory variables. The prediction is based on the value of the leaves that are in the same subgroup. Figure 21 shows an example using a tree-based model to determine market volatility level. The rules and conclusions in this example are straightforward and may not need any data to support them. For a tree-based model where the rules are learned from data, it becomes more complicated.

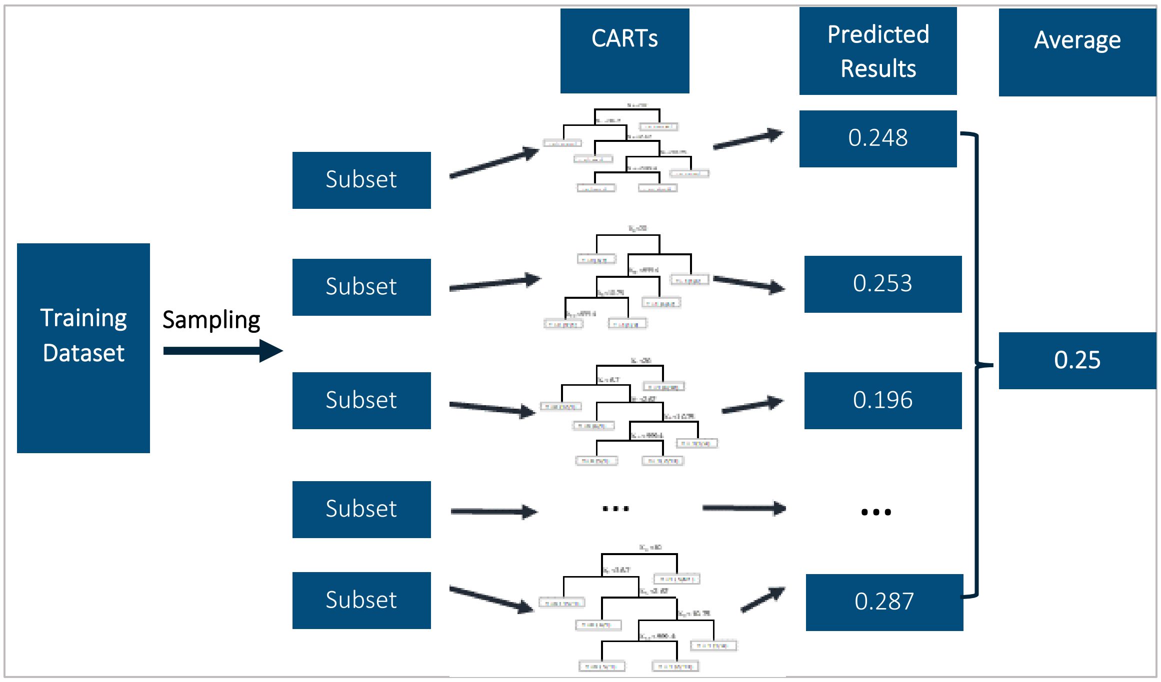
Figure 21
SAMPLE TREE-BASED MODEL



Classification and Regression Tree (CART) models are a basic form of tree-based models. CART models build trees to split the data based on explanatory variables. At each split, a variable is used to separate the data into two subgroups. The variable is chosen to provide the best split that improves the purity of the data in the subgroups.

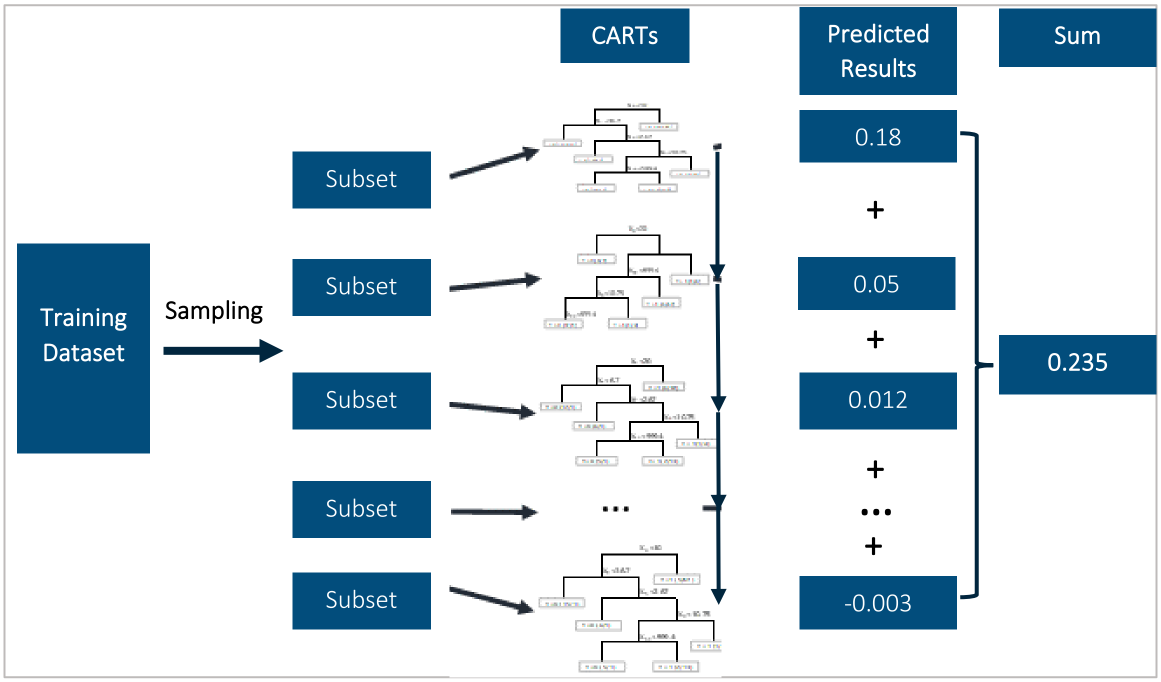
More advanced tree-based models are built upon CART. The famous **Random Forests models** are a random version of the CART models. Multiple subsets are sampled from the training dataset and each subset is used to build a CART model. Explanatory variables are sampled as well so that the relationship between the response variable and the explanatory variables will not be dominated by the most important ones. Less important explanatory variables can contribute to the final prediction as well. Figure 22 illustrates the structure of the Random Forests models used in this report. The final prediction is calculated as the average prediction by individual CART models.

Figure 22
RANDOM FORESTS MODEL STRUCTURE



Gradient boosting machine (GBM) is another decision tree–based ensemble method. Each tree is a weak estimator trying to estimate the residual error that the estimation of previous trees has caused. Gradually with a sufficient number of decision trees, the estimation error will decline to a very low level. Unlike Random Forests models which use parallel trees to predict in aggregate, GBM is a sequential tree model with the final prediction as the sum of predictions of all sequential trees, as shown in Figure 23.

Figure 23
GBM MODEL STRUCTURE



In this illustration, we have a target Y variable with a value of 0.24. Using a standard GBM, we first fit a CART model using a subset of data and a subset of features (explanatory variables). This first CART model will give us a predicted value of 0.18. The remaining difference is 0.06, calculated as the difference between 0.24 and 0.18. We then fit another CART model to the difference of 0.06 and get an estimation of the 0.05. And this process keeps going until the difference is small enough, or there is no further improvement of the prediction.

Model Training and Validation

The model training process tries to minimize the error of prediction based on the training dataset. The error is defined as the difference between actual value y_{actual} and predicted value y_{pred} based on **Root-mean-squared error** (RMSE): the square root of the mean of the square of all of the errors. Other definitions of errors can also be used as can combinations of errors.

$$RMSE = \sqrt{MSE} = \sqrt{\sum_{i=1}^{N} \frac{\left(y_{pred,i} - y_{actual,i}\right)^2}{N}}$$

After the model training process to minimize the error function, calibrated models need to be assessed and compared using standard validation methods. It is important to know that validation data (out-of-sample data) needs to be used for a meaningful comparison so that the issue of overfitting can be identified.

To assess the goodness-of-fit of regression models, other measures can be used beyond the RMSE metric which was part of the fitting procedure. A common alternative measure is coefficient of determination, also known as R^2.

$$R^2 = 1 - \frac{Unexplained\ Variation}{Total\ Variation} = 1 - \frac{\sum_i \left(y_{pred,i} - y_{actual,i}\right)^2}{\sum_i \left(y_{actual,i} - \bar{y}_{actual}\right)^2}$$

A high R^2 means that the model has a high accuracy to explain the variation of the target variable. Table 15 shows both the RMSE and R^2 using the training data and the validation data. Based on the out-of-sample data, most models have a high prediction accuracy, with Random Forests and GBM giving an R^2 higher than 94%.

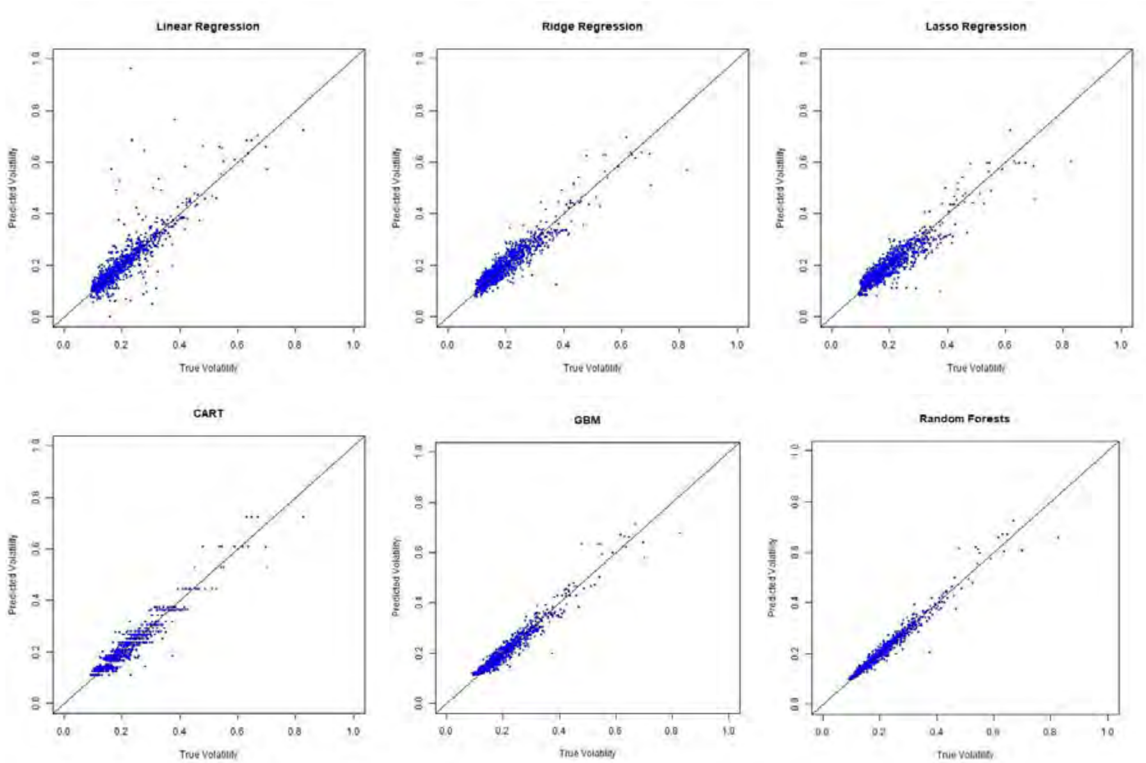
Table 15
GOODNESS-OF-FIT RESULTS

Model	In-the-sample data		Out-of-sample data	
	RMSE	R^2	RMSE	R^2
Linear Regression	0.0179	95.9%	0.0550	60.2%
Ridge Regression	0.0281	89.8%	0.0316	86.9%
Lasso Regression	0.0337	85.3%	0.0343	84.5%
CART	0.0226	93.4%	0.0252	91.6%
GBM	0.0179	95.9%	0.0212	94.1%
Random Forests	0.0061	99.5%	0.0164	96.5%

Models can be ranked based on goodness-of-fit measures at a high level. However, further analysis is usually desired to look at the actual predictions. Scatter plots of the actual values and predicted values are a good way to identify outliers and get comfortable with model accuracy. Figure 24 shows an example of a scatter plot to evaluate regression model accuracy. Dots lying on line y=x represent perfect estimation. Even if a model has a high R^2, scatter plots may help identify outliers which may be too important to ignore and may lead to a different model choice. The scatter plots of the CART model have discrete predicted values. The CART models use subgroup average as the estimate for any members in that subgroup. The Lasso, GBM, and Random Forests models show the least volatility around line y=x, which is consistent with the highest R^2 it has in this example. All models have some underestimation of the outlier on the right with the highest volatility in the out-of-sample data. Random Forests and GBM models have the smallest degree of underestimation.

Figure 24

SCATTER PLOT: REGRESSION MODEL VALIDATION



Considering both R^2 and the underestimation of high volatility cases, GBM and Random Forests may be chosen as the best models to explain the market volatility.

Key Findings and Approaches:

1. Predictive models are used to understand the contribution of each explanatory variable to the market volatility. The attribution analysis is not necessarily a cause-and-effect analysis. This is possible in scientific research such as vaccine effectiveness but is extremely difficult to apply to economic analysis because only one path of reality can be observed.
2. Daily frequency is used to make sure temporal relationships can be evaluated. Explanatory variables during previous periods are used to explain current market volatility. This temporal precedence ensures that the cause and effect are not swapped.
3. The predictive modeling process contains data processing, model training, and model validation. Using linear regression and tree-based models such as Random Forests and gradient boosting machines, the best calibrated model can explain 96% of variation in market volatility with satisfactory model valuation results based on scatter plots.

4.4 KEY FACTORS

With the models producing satisfactory accuracy using out-of-sample data, it is helpful to understand what explanatory variables are driving the prediction of high volatility. Each explanatory variable is assigned a feature importance score to indicate its contribution to the predictions, with the details explained in Appendix C: Feature Importance.

Before looking at individual explanatory variables, Table 16 shows the contribution of different data categories to the predictions and Table 17 shows the contribution of variables with different time lags to the predictions. Three model types are selected including GBM and Random Forests with the highest overall prediction accuracy and the Ridge regression which has the highest accuracy among linear models.

Table 16
CONTRIBUTION BY DATA CATEGORY

Model Type	Economic Data	Event Data	Investment style data	Retail investor data
Random Forests	20.8%	23.5%	43.1%	12.7%
GBM	20.4%	24.5%	45.7%	9.3%
Ridge Regression	18.4%	7.5%	14.9%	59.1%

Table 17
CONTRIBUTION BY TIME LAG

Model Type	1-month lag	2-week lag	1-week lag	3-day lag	1-day lag
Random Forests	39.0%	17.9%	11.4%	12.8%	18.9%
GBM	50.0%	6.3%	5.9%	6.7%	31.2%
Ridge Regression	22.4%	18.1%	16.7%	18.0%	24.8%

Looking at the two most accurate model types in this analysis, retail investor data contributed about 11% of the variation in the market volatility. In aggregate, the explanatory variables have high predicting power for the next trading day (1–day lag) and next month (1–month lag).

Looking at individual explanatory variables, Table 18 lists the top 20 variables by model type. For the two tree-based models, the 20 variables take more than 70% of the total importance of all explanatory variables.

Table 18
TOP 20 IMPORTANT FEATURES

Random Forests			GBM			Ridge Regression		
Variable	Lag	Importance[1]	Variable	Lag	Importance	Variable	Lag	Importance
AUM_ELS	1m	1.000	AUM_fof	1m	1.000	BAA10Y	1d	1.000
AUM_fof	1m	0.876	AUM_ELS	1m	0.981	BAA10Y	3d	0.474
T5YIE	1d	0.336	BAA10Y	1d	0.534	AUM_ELB	1m	0.462
AUM_ELS	2w	0.277	covid_death_us	1d	0.331	DGS1	1d	0.453
AUM_fof	2w	0.275	ma_debit	1w	0.206	AUM_EMEE	1m	0.379
AUM_ELB	1m	0.271	gt_covid	1d	0.161	UMCSENT	1d	0.344
covid_death_us	1d	0.242	W994RC1Q027SBEA	1m	0.125	DGS10	1w	0.341
BAA10Y	1d	0.227	ma_debit	1d	0.124	DGS10	2w	0.341
covid_death_us	1w	0.221	covid_death_us	1m	0.116	W994RC1Q027SBEA	1d	0.335
covid_death_us	1m	0.204	GPDI	1d	0.108	AUM_ED	1m	0.332
covid_death_us	2w	0.203	Reddit_job	1m	0.103	AUM_fof	1m	0.326
covid_death_us	3d	0.189	T5YIE	1d	0.092	DGS10	1d	0.307
AUM_HF	1m	0.183	ma_debit	3d	0.077	AUM_ELO	1m	0.304
BAA10Y	3d	0.149	AUM_HF	1m	0.075	AUM_Con	1m	0.288
ma_debit	1w	0.125	AUM_MS	1m	0.061	AUM_EMLA	1m	0.281
ma_debit	3d	0.121	gt_covid	1m	0.055	AUM_fof	1d	0.273
Reddit_job	1d	0.094	Reddit_bull	1d	0.055	ma_debit	1d	0.271
Reddit_job	1m	0.093	gt_stock_market	1d	0.055	RH_AUC	1d	0.256
W994RC1Q027SBEA[2]	1m	0.088	Retail_Share	1d	0.049	CPIAUCSL	1d	0.252
ma_debit	1d	0.088	Reddit_would	3d	0.041	DGS10	1m	0.249
71.4% of total importance			**83.5% of total importance**			**13.3% of total importance**		

Notes:

1. *Importance scores are normalized so that the maximum score is 1.*
2. *W994RC1Q027SBEA is the private net lending or net borrowing.*

Key Findings:

1. Economic data, event data, investment style data, and retail investor data contributed to explaining the market volatility collectively. For example, based on the calibrated Random Forests model, retail investor data contributed more than 12% of the variation in the market volatility.
2. In aggregate, the explanatory variables have good predicting power for not only the volatility of next trading day (1–day lag), but also next month (1–month lag).
3. For the two tree-based models with highest predicting accuracy, the top 20 variables take more than 70% of the total importance of all explanatory variables.

Section 5: Practical Implications

The attribution analysis showed that economic data, event data, investment style data, and retail investor data all contributed to the heightened market volatility in the post-2020 period. For each category, the situation can be examined separately for each data category to assess whether it will continue in the future. This can then be used to determine how much of the observed changes in the recent period may be reflected in the future assumptions.

5.1 ECONOMIC ASSUMPTION

The attribution analysis provides the modeling framework that can be used to estimate future volatility based on the current conditions. In addition to updating short-term conditional volatilities, it can also be used to assess the impact on medium- tolong-term volatility assumptions. A simple method is to assess whether each category of explanatory variables will remain their current status or patterns in the considered time horizon. A credibility score can be assigned to each category after the assessment. As shown in Table 19 as an example, if the difference between the current volatility level and existing assumption is 5%, the adjustment to the volatility assumptions may be determined as the weighted average of creditability score and share of the difference based on the contributions.

Table 19
EXAMPLE: VOLATILITY ASSUMPTION ADJUSTMENT

Category	Contribution[1]	Credibility[2]	Volatility Impact[3]
Economic	20.60%	0.1	0.10%
Event	24.00%	0.1	0.12%
Investment style	44.40%	0.2	0.44%
Retail investor	11.00%	0.8	0.44%
Total			1.11%

Notes:

1. *The contribution is determined as the average of Random Forests and GBM results in Table 16.*
2. *Credibility scores may be determined based on forecasts of explanatory variables. If a variable is forecasted to return to normal soon, a low credibility can be assigned.*
3. *The volatility impact is calculated as Contribution X Credibility X (Current Volatility – Volatility Assumption). For example, the volatility impact of retail investor factors is calculated as 11% X 0.8 X 5%, which equals 0.44%.*

In addition to the market volatility assumption, the observations during the post-2020 period may also lead to reassessment of modeling choices.

1. Modeling frequency. Modeling frequency can have a significant impact on results of financial projection. Models with low frequency such as annual and quarterly frequency may underestimate the risk exposure significantly and the short-term impact of extreme events. Fewer historical data points are available for calibrating the models as well. Models with a higher frequency such as the daily frequency may be more appropriate in a volatile period to be able to capture autocorrelations and cross correlations among different variables. A higher frequency also gives us the flexibility to evaluate risks at a lower frequency without losing the important details. For example, daily equity index returns can be transformed to monthly, quarterly, or annual returns.

2. Reflection of non-constant volatility. Models with constant volatility or fixed volatility term structure may be replaced with stochastic volatility models that reflect volatility of volatility and volatility clustering.

3. Number of scenarios to use. The sampling errors increase with a higher volatility level, a higher degree of volatility clustering and/or the existence of discrete jumps. A larger set of scenarios may be needed to maintain the same level of convergence when calculating risk measures such as value at risk and tail value at risk using real-world scenarios, or even the fair market value of liability cashflows with embedded options and guarantees using risk-neutral scenarios, if the magnitude of volatility increase is material.

4. The significance of outliers. When outliers cannot be explained by stochastic volatility models, jump diffusion models that contain discrete jumps may be used to reflect the extreme events.

5. Nonlinearity and temporal relationships. With higher correlations observed in most extreme events, nonlinear relationships need to be reflected through methods such as state-dependent correlation matrices or copulas. When using models with high frequency, temporal relationships also need to be incorporated in the models through autocorrelations and cross correlations.

Key Findings:

1. The attribution analysis provides the modeling framework that can be used to estimate future volatility based on the current conditions. A simple method is to assess whether each category of explanatory variables will retain their current status or patterns in the considered time horizon.
2. In addition to the market volatility assumption, the observations during the post-2020 period may also lead to reassessment of modeling choices, including modeling frequency, reflection of non-constant volatility, number of scenarios for stochastic analysis, using jump diffusion models, and nonlinear and temporal relationships.

5.2 ASSEST ALLOCATION OPTIMIZATION

For some of the suggested changes in the previous section, an example of liability driven investment optimization is used to illustrate the potential impacts of volatility assumption and model changes. Table 20 lists the major assumptions used in the example, with more details available in Appendix A.5.

Table 20
ASSET ALLOCATION EXAMPLE SPECIFICATION

Item	Specification
Asset classes	Public equity • Annual expected return: 13% • Annual volatility: 22% Bond fund that matches liability duration • Annual expected return: 5% • Annual volatility: 8% Correlation between equity and bond fund returns: 0.045
Liability	Liability values are projected using bond fund returns given duration matching strategy with a 1% tracking error. Benefit payments are assumed to be 2% of initial lability value.
Economic scenario generators	Three ESGs are used to generate economic scenarios to assess investment strategies: • Geometric Brownian Motion model with constant volatility model • Stochastic volatility model • Stochastic volatility with jump diffusion model
Optimization criteria	By simulating future asset and liability values based on economic scenarios, asset/liability (funding ratio) is projected under each scenario at a chosen time horizon. An aggregated score of the ending funding ratios is used to represent the outcome of a chosen asset allocation plan. The score is determined using reference-dependent utility function in Warren (2019)
Testing Scenarios	Initial funding ratios: 70% to 140% with an incremental step of 2.5% Time horizon: 3-year horizon Asset allocation plans: 0% to 100% equity investment with an incremental step of 1%

As listed in Table 21, step-by-step changes in the market volatility assumption and models are made to show the potential impact on asset allocation. These changes may be justified by the observations in post-2020 period.

Table 21
MOVEMENT ANALYSIS STEPS

Step	Description	Detail
1	Baseline	• Equity Volatility: 22% • Constant volatility • Annual frequency • 5,000 scenarios
2	Increased volatility	• **Equity Volatility: 25%** • Constant volatility • Annual frequency • 5,000 scenarios
3	Higher frequency (returns are generated daily and accumulated to annual frequency)	• Equity Volatility: 25% • Constant volatility • **Daily frequency** • 5,000 scenarios
4	Volatility clustering	• Equity Volatility: 25% • **Stochastic volatility** • Daily frequency • 5,000 scenarios
5	Jumps in returns	• Equity Volatility: 25% • **Stochastic volatility with jumps** • Daily frequency • 5,000 scenarios

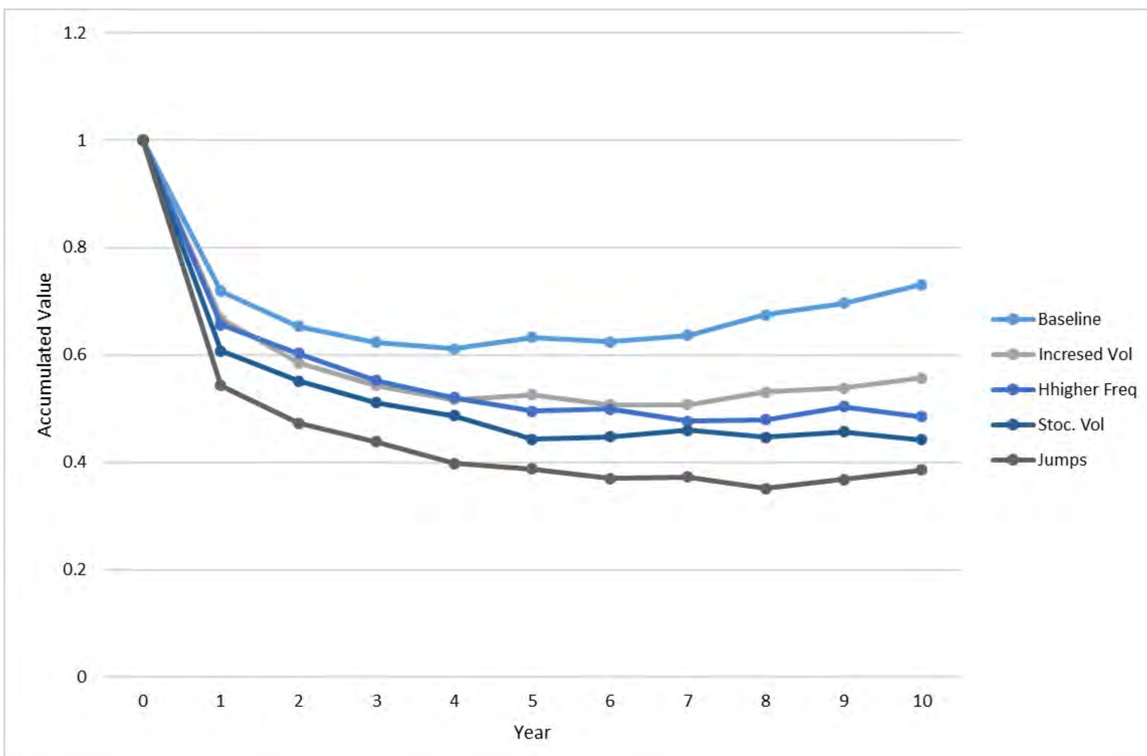
As shown in Table 22, except from the baseline to the increased volatility, average equity returns and volatilities stay at the same level for changes in the modeling approach. The range did change because of the inclusion of higher frequency, volatility clustering and jumps in the equity returns.

Table 22
STATISTICS OF GENERATED SCENARIOS

Step	Description	Average equity return	Equity volatility	Range	Average bond fund return	Bond volatility	Range
1	Baseline	0.13	0.22	[-0.51,1.36]	0.052	0.083	[-0.23,0.45]
2	Increased volatility	0.13	0.25	[-0.58,1.64]	0.052	0.083	[-0.23,0.45]
3	Higher frequency	0.13	0.25	[-0.61,1.69]	0.051	0.083	[-0.24,0.46]
4	Volatility clustering	0.13	0.25	[-0.69,1.31]	0.052	0.084	[-0.23,0.48]
5	Jumps in returns	0.13	0.25	[-0.8,1.41]	0.052	0.083	[-0.23,0.45]

Even though the market volatilities may be the same, the distribution of equity returns changes by the modeling approach, as indicated in the range of simulated equity returns in each step. Figure 25 shows that the 1st percentile of cumulative equity index values of the five steps, with material differences among different assumption and/or modeling approaches.

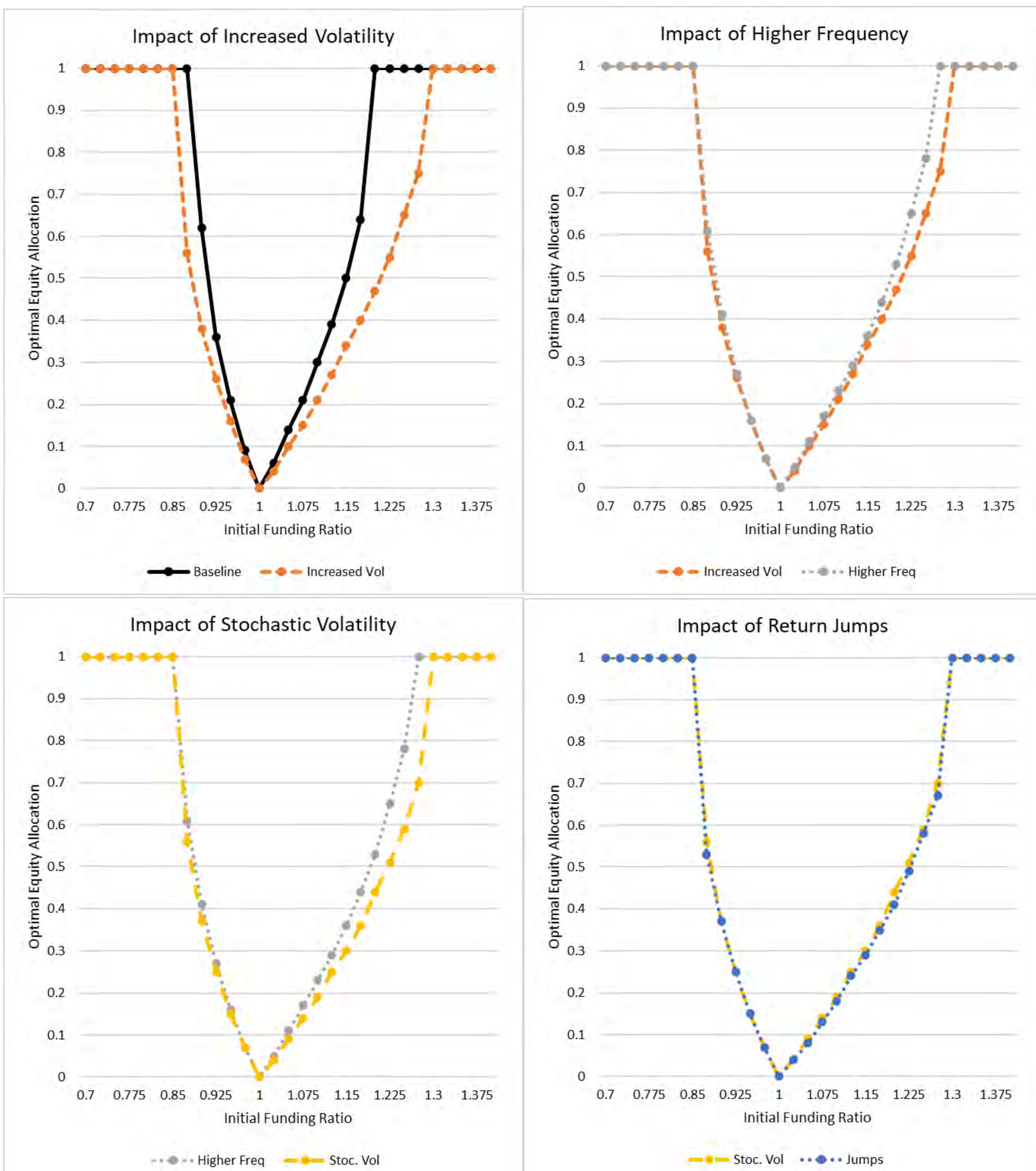
Figure 25
CUMULATIVE EQUITY INDEX VALUE: 1ST PERCENTILE



The impact of volatility assumption and modeling approaches can have material impact on the resulting optimal asset allocation plan, as shown in Figure 26. Increasing the equity volatility assumption causes reduction in optimal equity allocation, as expected. Using daily frequency leads to slightly higher equity allocations in the cases of overfunding. In this step, there is no volatility clustering modeled. Applying stochastic volatility reduces the optimal equity allocation in many cases, even though the general market volatility level stays the same. Adding return jumps has some marginal impact on equity allocations, with 0% to 3% reduction observed. In all cases, 0% equity allocation is always desired when the initial funding ratio is 100%, because it is assumed that the bond fund tracks the liability portfolio with small tracking errors and underfunding is penalized based on the utility function.

Figure 26

OPTIMAL ASSET ALLOCATION PLAN MOVEMENT ANALYSIS



This example illustrates the potential impact of applying changes observed in post-2020 to asset allocation. In practice, additional consideration such as capital requirements can be included for a more holistic analysis.

Key Findings:

Using a liability driven investment optimization example, the impact of volatility assumption and modeling approaches can have material impact on the resulting optimal asset allocation plan.

- Increasing the equity volatility assumption causes reduction in optimal equity allocation.
- Applying stochastic volatility reduces the optimal equity allocation in many cases, even though the general market volatility level stays the same.
- Adding return jumps has some marginal impact on equity allocations, with 0% to 3% reduction observed.

5.3 OTHER IMPLICATIONS

If the suggested changes in Section 5.1 are adopted in financial modeling to make informed business decision-making, in addition to their impact on asset allocation, many other areas will be affected as well.

1. Liability valuation and capital requirement. The distribution of financial outcomes is likely to have larger ranges, and more importantly, heavier tails, given that the assets backing liability may be more volatile and the underlying assets that the guaranteed liability value is linked to may be more volatile. For liability valuation and capital management that use high confidence levels, increases in the liability values and capital requirements are expected, ceteris paribus.
2. Hedging strategies that focus on first-order sensitivities such as Delta (sensitivity to equity) and Rho (sensitivity to interest rate) may see lower hedging effectiveness. In addition, increasing hedging costs during extreme events may make certain hedging programs too expensive to implement. Second-order sensitivities such as Gamma (sensitivity to Delta) and Vega (sensitivity to implied volatility) may need to be incorporated into hedging programs to be immune to stochastic volatilities. Dynamic hedging programs need to monitor these second-order sensitivities and adjust hedging positions in a timely manner. Financial derivatives on market implied volatility such as volatility swaps and options may be used more frequently to mitigate the risk of having volatile cost of first-order hedging. Hedging positions may be assessed and adjusted at least on a daily basis to reduce the impact of market illiquidity during extreme events.
3. The cost of providing guarantees of investment performance may be found too high. The guaranteed level may be lowered together with lower premium rates or higher upside potential. For guarantees that are backed with long-term asset liability matching strategies, appropriate penalty for early termination may be designed to offset the cost of asset and liability mismatch. Effective communication with policyholders is also important to manage their expectation and behaviors to mitigate the exposure to heightened volatility risk.
4. The risk-absorbing capability may be reassessed given new volatility assumptions. Investment risk may be shared with the capital market using reinsurance, structured instruments, and financial products that the payments are linked with capital adequacy ratio.

Even though volatility assumptions may not be changed immediately, given the possibility that recent market volatility behaviors may have long-term impact based on the findings from the attribution analysis, it is beneficial to quantify the potential financial impact if volatility assumptions and models change and make contingent plans that may be triggered if market conditions such as conditional volatility reach a certain threshold.

Key Findings:

Many other areas can be affected by market volatility risk, in addition to asset allocations. Examples include liability valuation, capital management, hedging strategy, product offering on long-term guarantees, and risk mitigation plans.

Section 6: Further Developments

This research uses a data-driven approach to explore the factors that may have contributed to the elevated market volatility observed in the past few years. It can be extended to reflect more sophisticated patterns and solve other problems.

1. Although quantitative analysis may be able to present findings in an objective way, qualitative analysis can provide valuable insights for future given structural changes. The data-driven approach has an implicit assumption that the patterns in the historical data will persist in the future. Limited recent information may indicate changes in the pattern, but it may not change the prediction due to insufficient statistical credibility. Qualitative analysis focusing on recent potential structural changes such as monetary policies may be used to further improve the attribution analysis and short-term prediction.
2. Insurance companies also use alternative asset classes such as real estate, private equity, and commodities to diversify their portfolios and get exposure to certain asset classes. Their volatilities can be analyzed using methods similar to those applied to this research for public equity market volatilities. Data availability in terms of historical periods and frequency may be less for those alternative classes.
3. In economic and capital market analysis, only one scenario, the actual scenario, can be observed. Although leading indicators can be used in the attribution analysis, it is not a pure cause-and-effect analysis because we are not able to observe two or more scenarios in the same time period. Without the control groups, we do not know for sure if the leading indicators in the treatment groups were the real causes or just happened to occur. It may be beneficial to find control groups by identifying historical periods that are similar to the post-2020 period except the factor(s) to be evaluated, such as retail investor activities and pandemic events. The control groups may be selected from the history of other countries or constructed using sophisticated simulation models. Improved cause-and-effect analysis can reinforce the effectiveness of these leading indicators.
4. Plausible scenarios or stress scenarios may be constructed using the identified patterns and the possible chain of events, in addition to the general market volatility level. For example, will a potential resurgence of more deadly variants of the COVID virus lead to a long-lasting period of high market volatility? Will the inflation and injected liquidity cause a follow-up recession in the near future? The attribution analysis needs to be extended or adjusted to reflect the conditions in the what-if analysis.

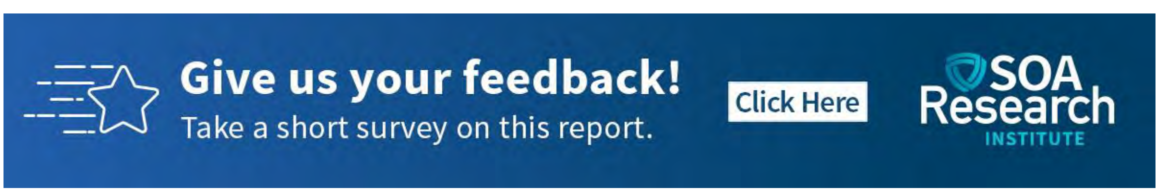
Section 7: Conclusion

With the increasing market uncertainty observed during and post the recent pandemic, an in-depth analysis of the market volatility can help us understand the key drivers and assess their long-term impact. The post-2020 period is one of the most volatile periods since 1928 in terms of realized volatility, implied volatility, volatility of volatility, and jumps, even though the economic recession only lasted for a couple of months. Autocorrelation was also more extreme compared to other historical extreme events which indicates a high degree of volatility clustering. Changes in both contemporary and temporal relationships are observed during the post-2020 period with equity index return more negatively correlated with market volatility, and high returns preceding higher volatility.

A variety of explanatory variables are used to perform attribution analysis, including economic data, retail investor data, investment style data, and event data. Data exploration based on cross correlations indicates many variables such as retail investor sentiment, real GDP growth rate, private investment, federal deficit, and investor leverage ratio are correlated with market volatility individually. The high correlation existed not only on concurrent data but also on lagged data where data of explanatory variables preceded market volatility data.

Using predictive modeling, as high as a 95% change in the market volatility can be explained by preceding explainable variables. Economic data, event data, investment style data, and retail investor data all played important roles to explain the conditional market volatility. The predictive models used in the attribution analysis can be used to predict the future conditional volatility and evaluate the potential impacts on economic assumptions, asset allocation, hedging strategies, and risk management. Forward-looking views can be incorporated in the values of explanatory variables to determine appropriate market volatility assumptions and modeling choices such as model frequency, stochastic volatility with jumps, and nonlinear contemporary and temporal relationships.

Changes in volatility assumption and models may affect many areas including optimal asset allocation, liability valuation capital requirement, hedging strategy, product design and management, and risk mitigation plans. It is helpful to quantify their potential financial impacts to allow informed decision-making and contingent planning.



Section 8: Acknowledgments

The authors would like to thank all members of the Project Oversight Group (POG) tasked with providing governance on this research project. This paper would not have attained its current level of relevance to practitioners without the POG's guidance, feedback and insightful input.

"Market Volatility Risk Management in an Era of Extreme Events" POG members are the following:

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- Kimberly Gordon
- Antony Moyalan, ASA
- Robert Reitano, FSA, MAAA, CERA
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- Feng Sun, FSA, CERA
- Fabien Thonar, FSA
- Tianyang Wang, ASA

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Appendix A: Market Volatility Analysis

This appendix is used to provide additional information to support Section 2.

A.1 NASDAQ DATA ANALYSIS

Figure A.1

NASDAQ DAILY RETURNS AND RETURN VOLATILITIES

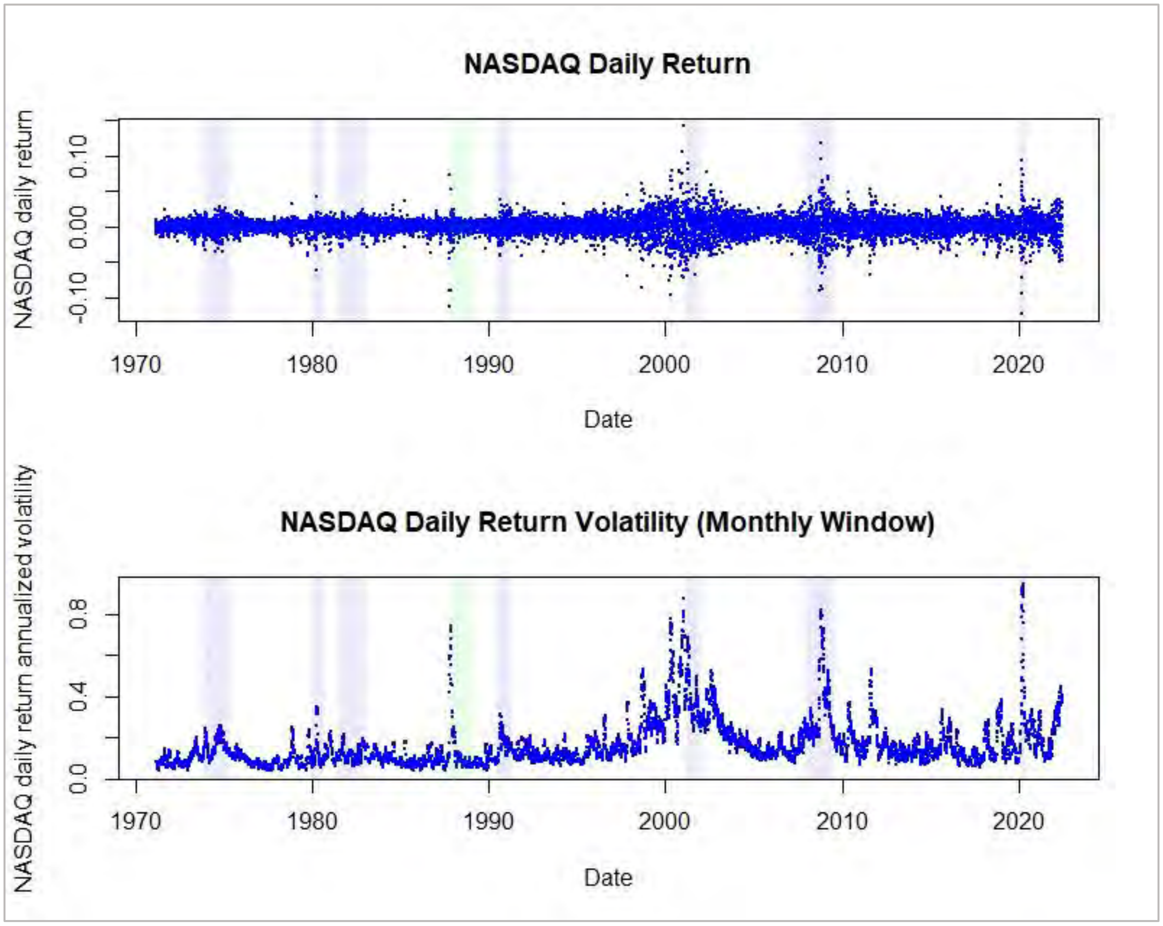


Figure A.2
NASDAQ IMPLIED VOLATILITIES

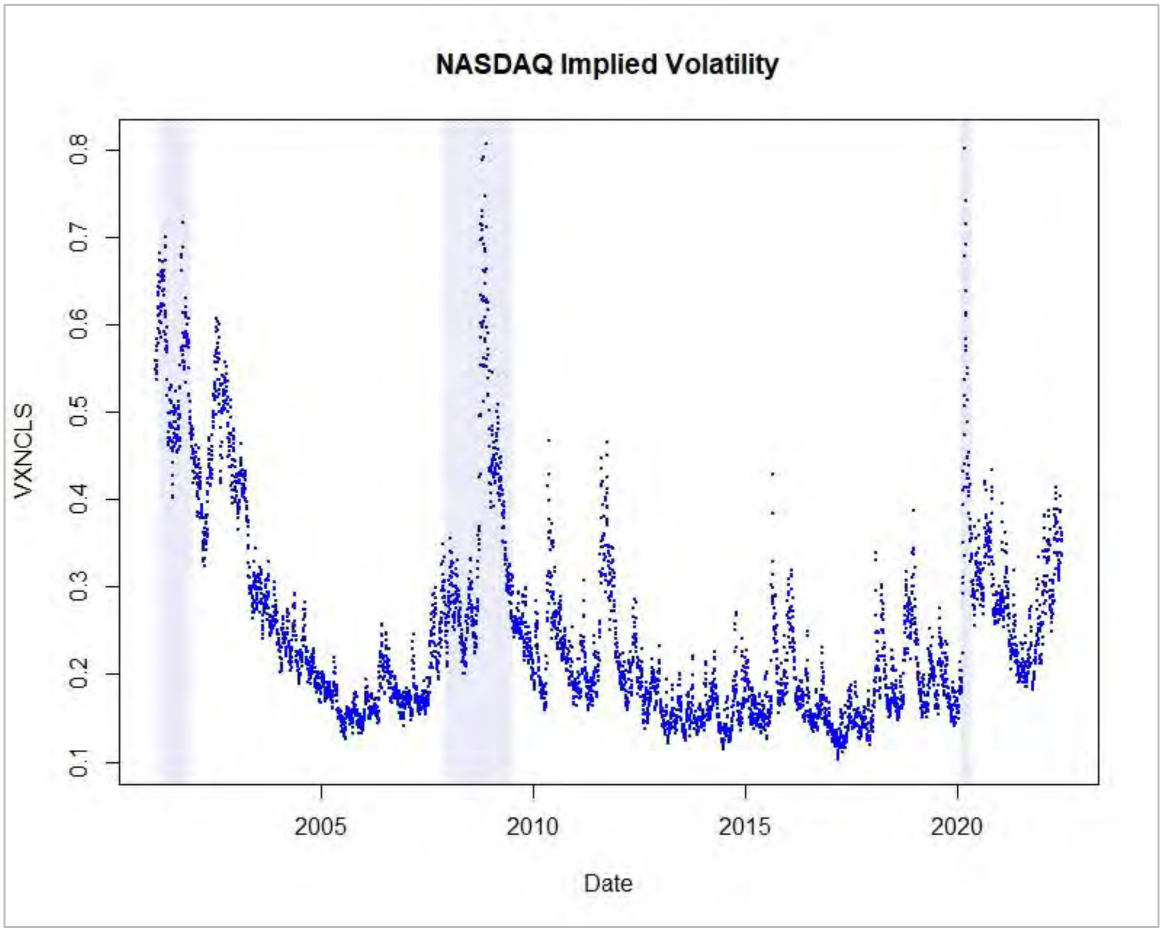
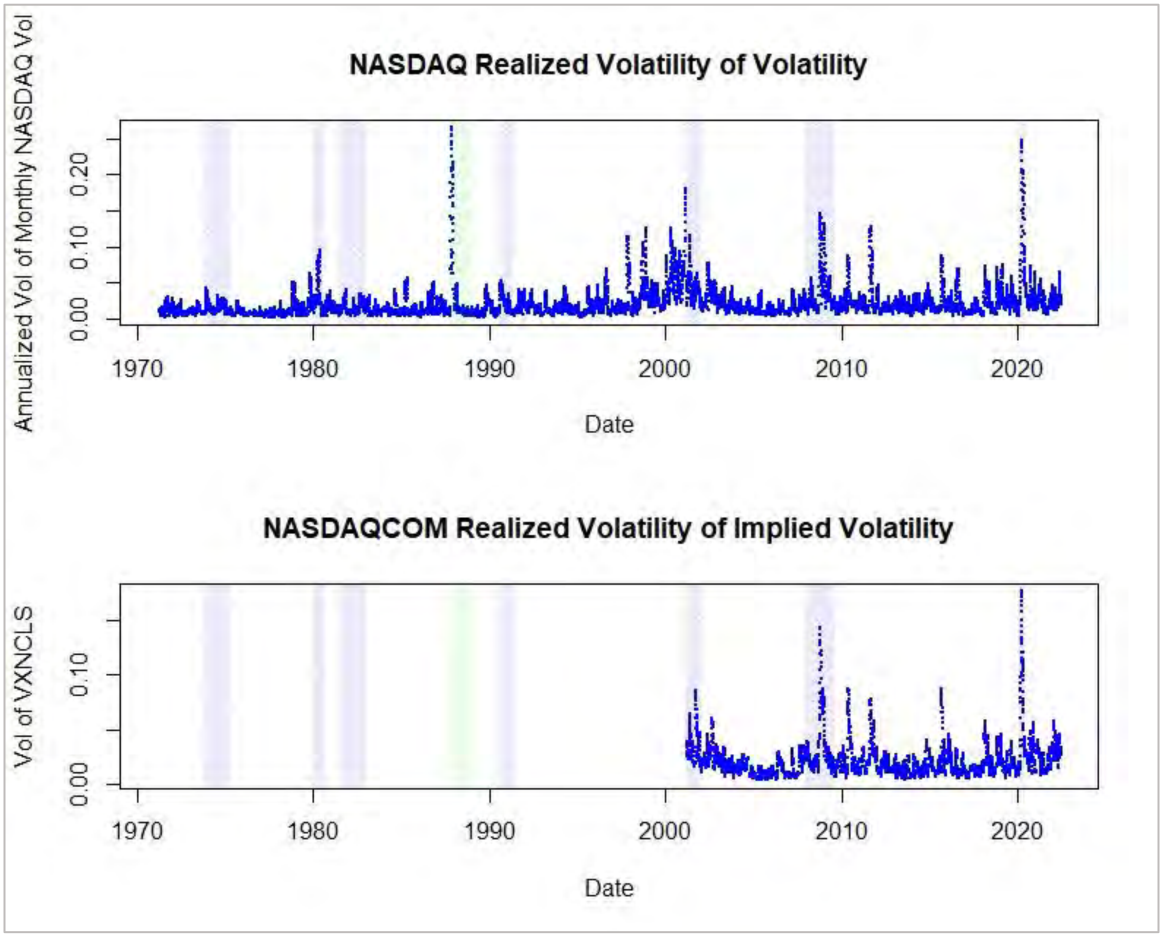


Figure A.3
NASDAQ REALIZED VOLATILITY OF VOLATILITIES



A.2 RUSSELL 2000 DATA ANALYSIS

Figure A.4
RUSSELL 2000 DAILY RETURNS AND RETURN VOLATILITIES

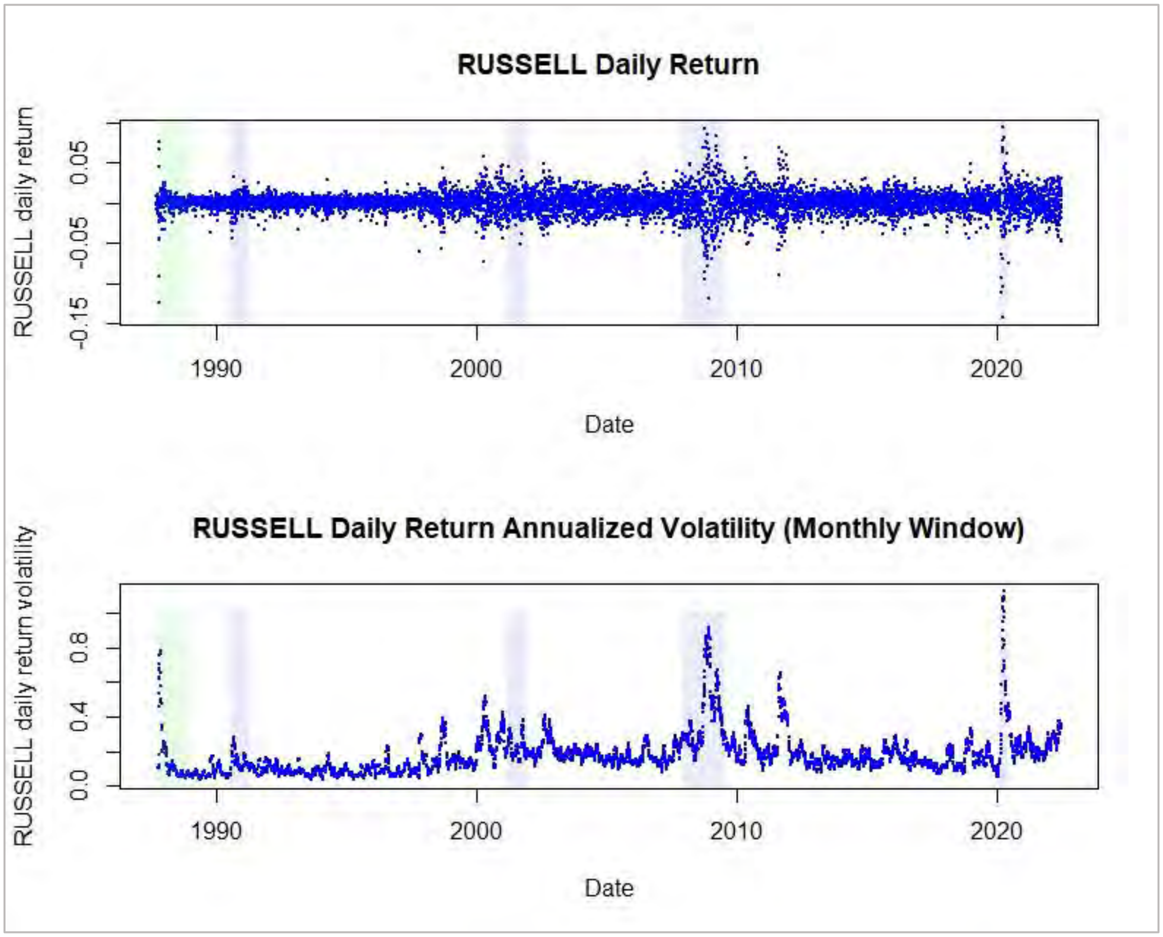


Figure A.5
RUSSELL 2000 IMPLIED VOLATILITIES

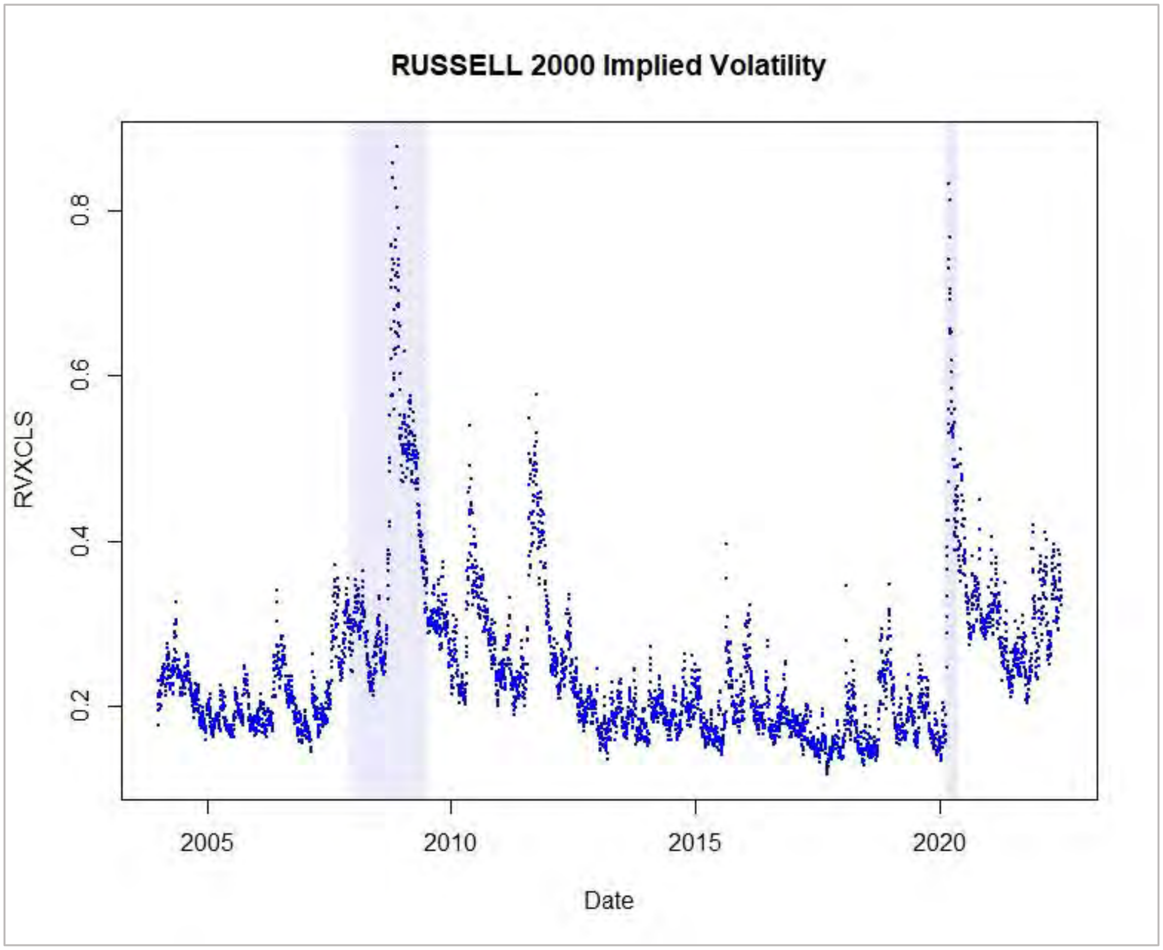
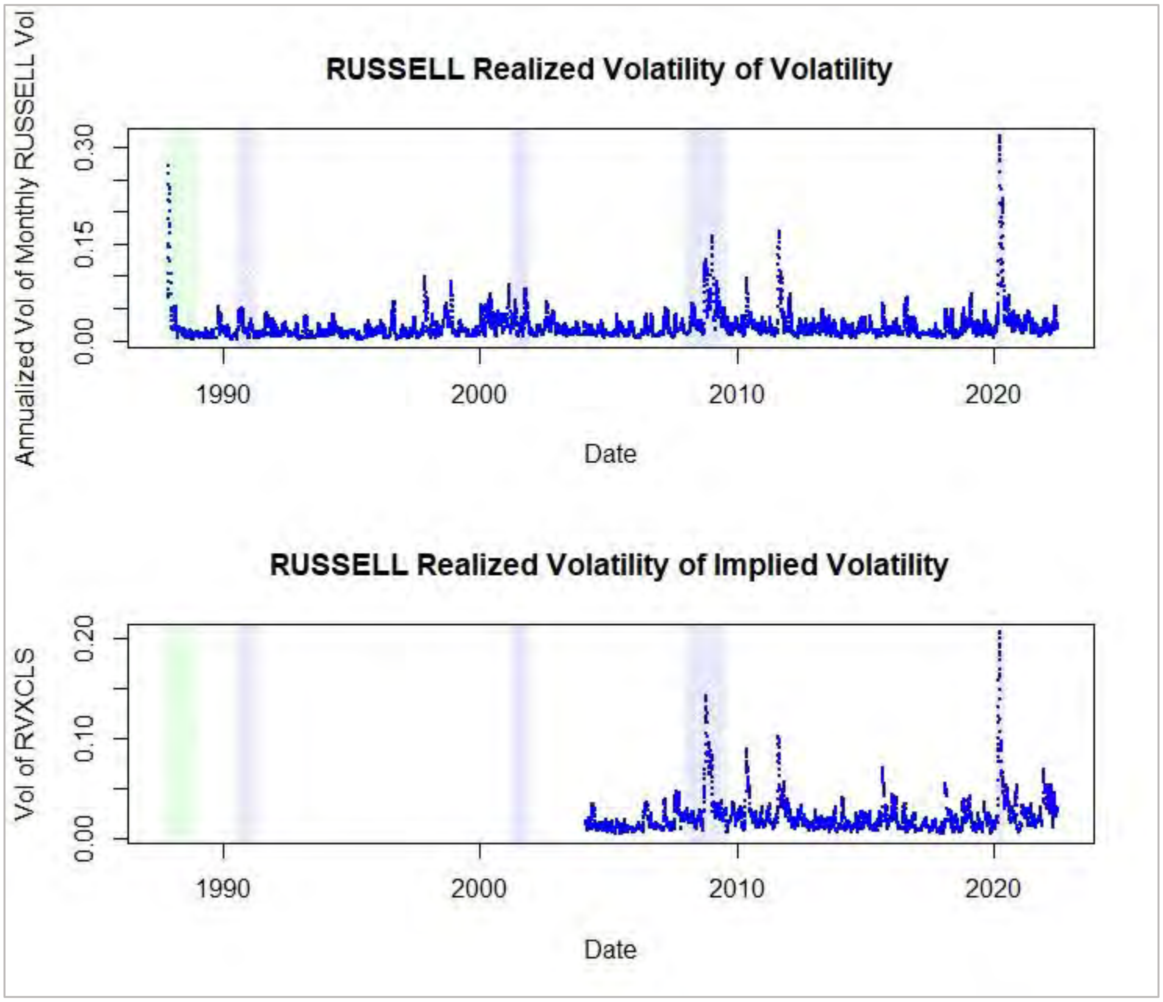


Figure A.6
RUSSELL 2000 REALIZED VOLATILITY OF VOLATILITIES



A.3 STL ANALYSIS

Seasonality and trend analysis using Loess (STL) is performed for four extreme periods: the great depression, 1987 Black Monday, the 2008 financial crisis, and the recent COVID pandemic. A minimum of two years is used for the studied periods. By comparing the residuals, the recent period has volatilities at similar level to other extreme periods.

Figure A.7

STL ANALYSIS: GREAT DEPRESSION

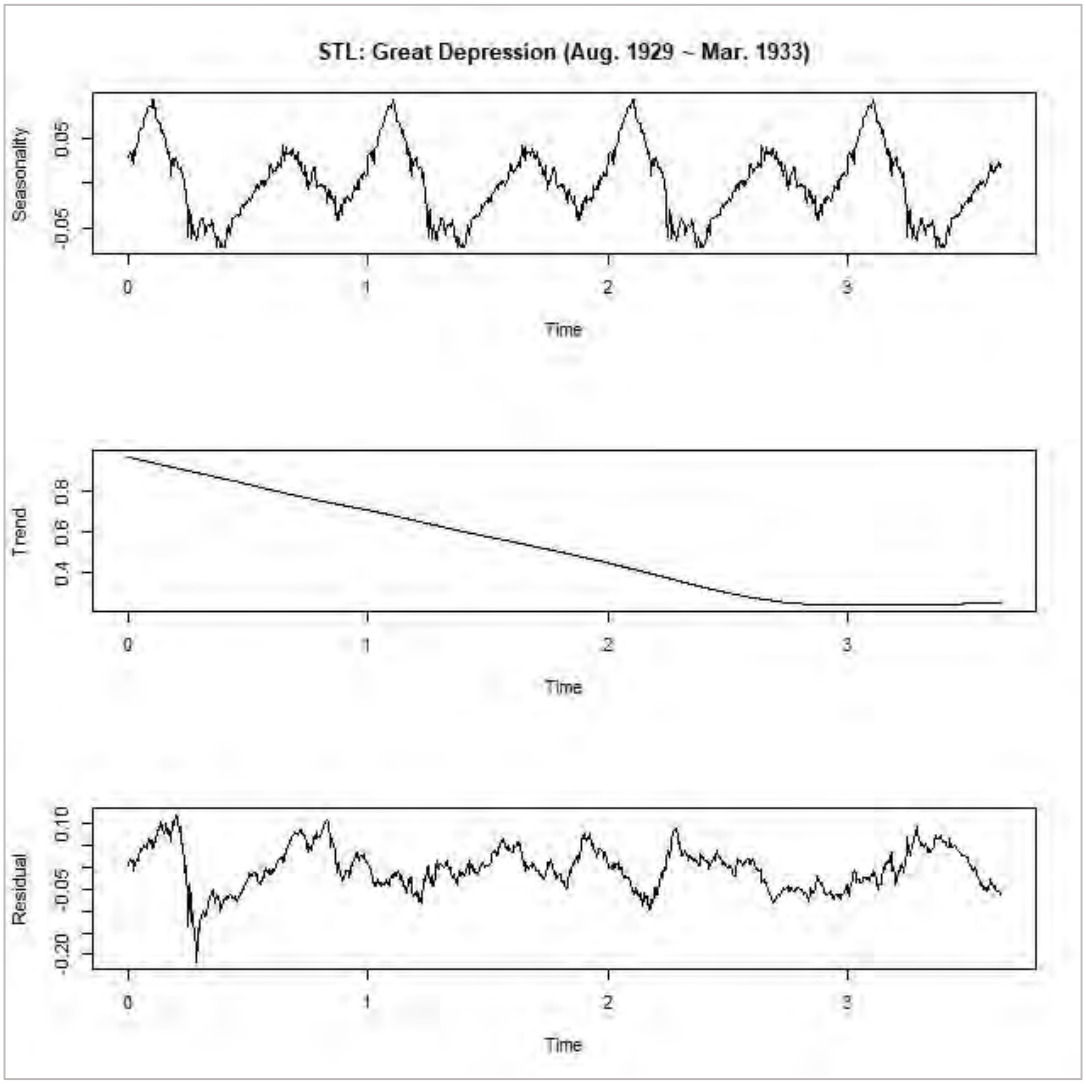


Figure A.8

STL ANALYSIS: 1987 BLACK MONDAY

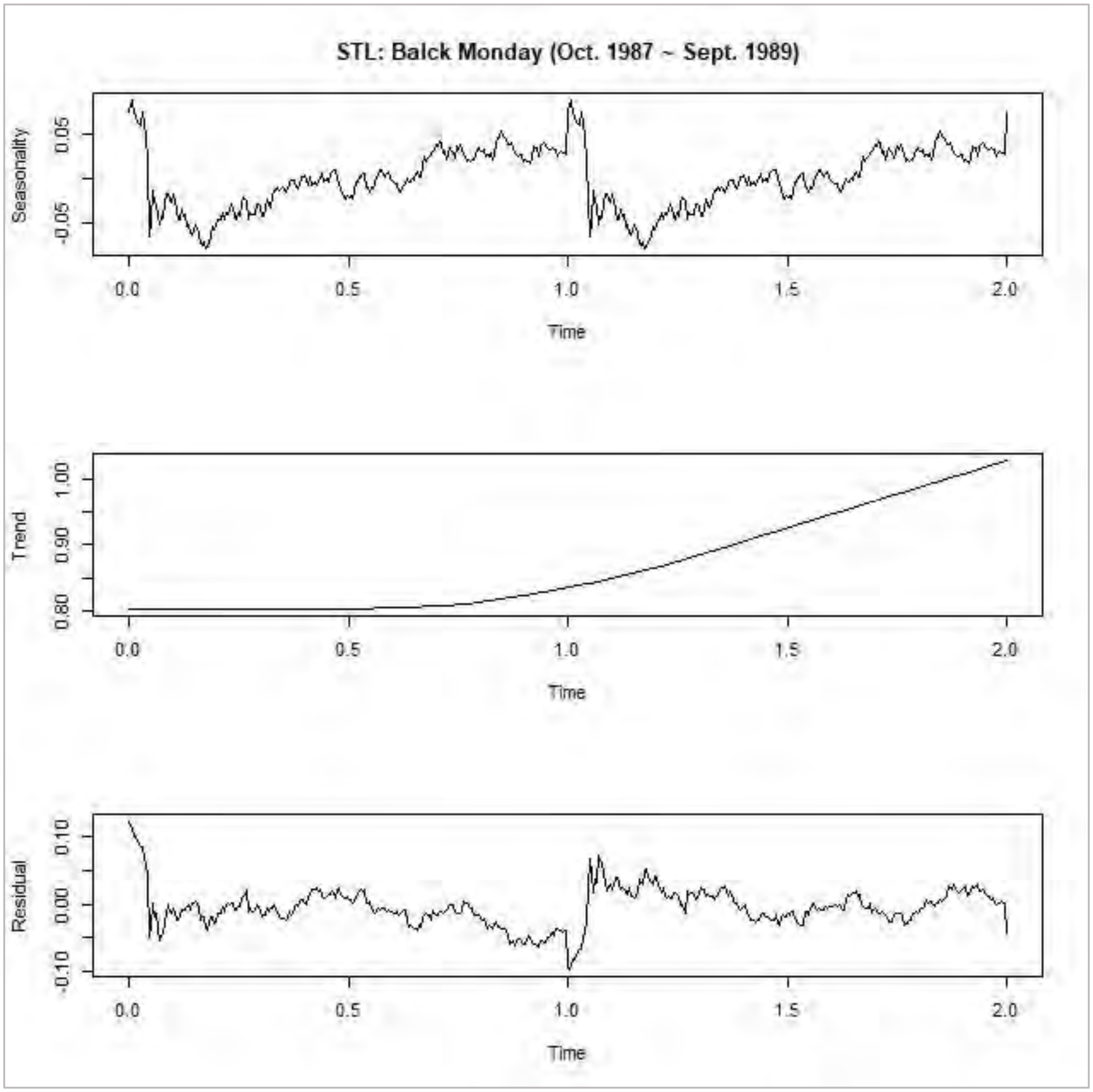


Figure A.9

STL ANALYSIS: 2008 FINANCIAL CRISIS

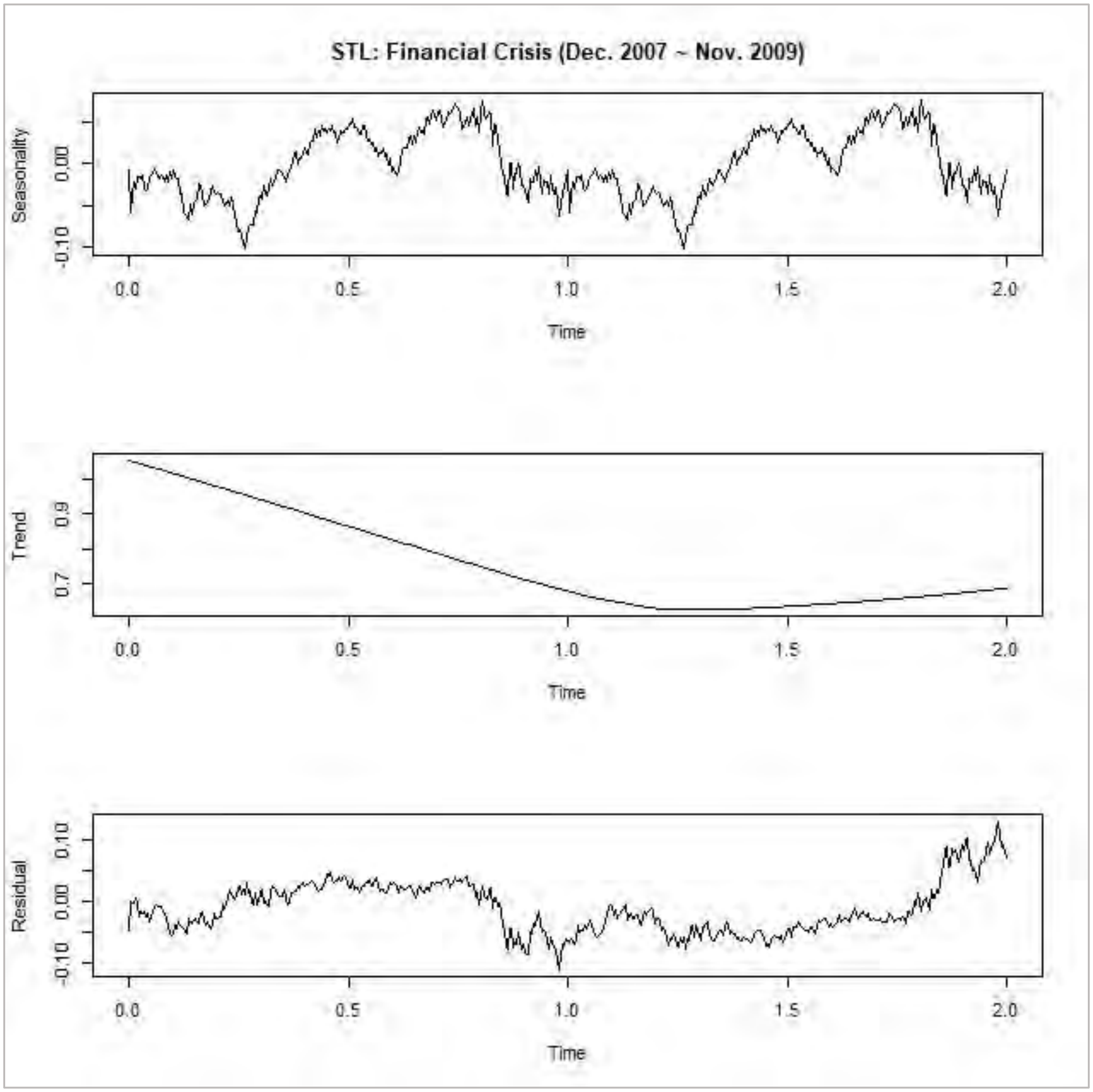
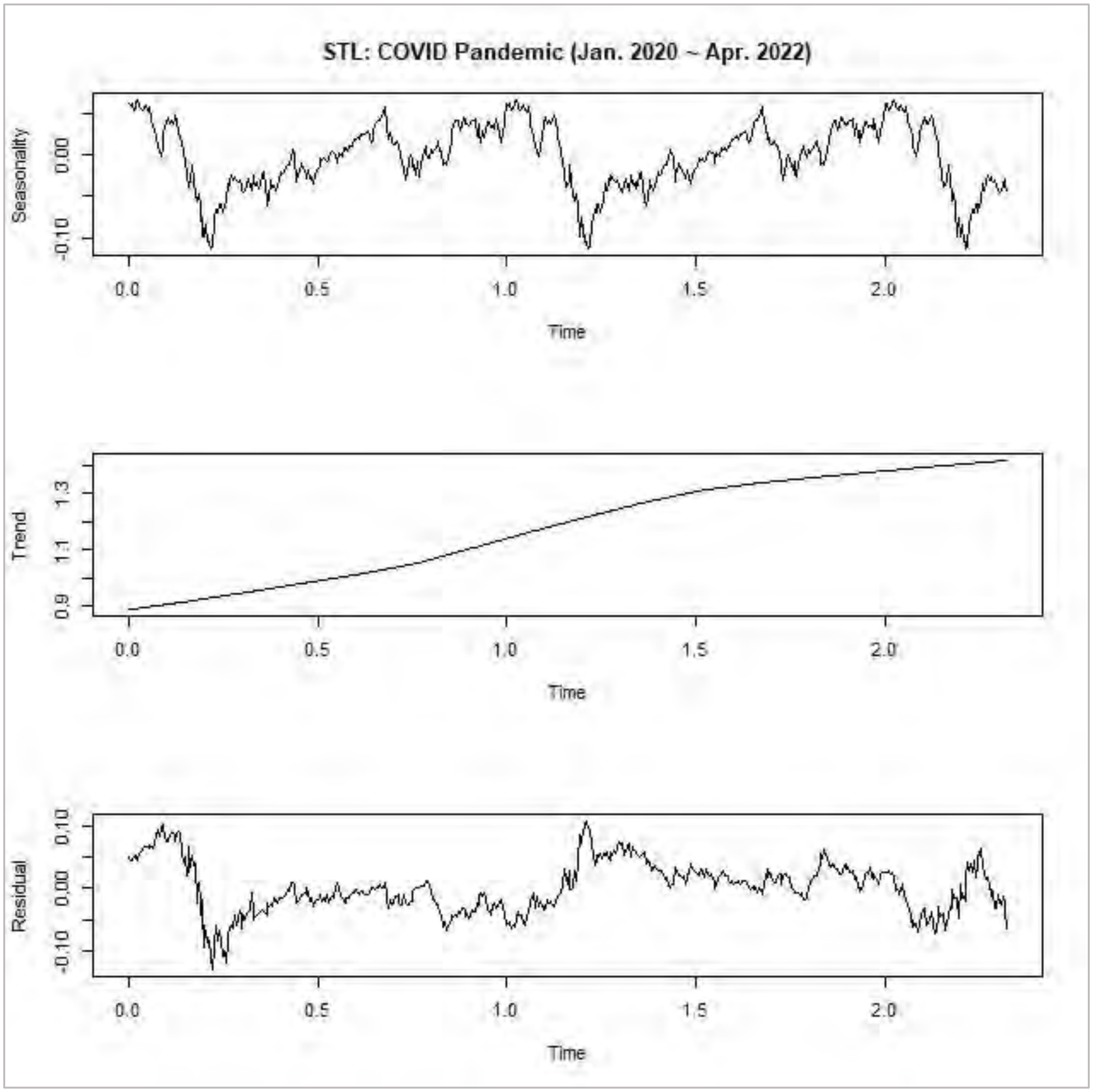


Figure A.10
STL ANALYSIS: COVID PANDEMIC



A.4 VECTOR AUTOREGRESSIVE MODEL PARAMETERS

Table A.1

VAR MODEL PARAMETER (JAN 2020 — JUN 2022)

Parameters	SPX_rtn	VIXCLS	T10YIE	DFF	DGS1	DGS10	BAA10Y
SPX_rtn.l1	-0.396	0.544	-0.003	-0.001	-0.006	-0.004	-0.001
VIXCLS.l1	-0.068	0.963	-0.004	0.000	-0.003	0.000	0.001
T10YIE.l1	-0.649	-1.264	1.042	0.080	-0.018	-0.105	-0.081
DFF.l1	-2.092	0.993	-0.033	0.927	-0.090	-0.218	-0.020
DGS1.l1	-2.593	2.743	0.119	-0.281	1.117	0.052	0.010
DGS10.l1	-0.237	-1.324	-0.099	-0.030	0.040	1.083	0.052
BAA10Y.l1	-2.233	0.455	-0.034	-0.054	0.034	0.070	1.042
SPX_rtn.l2	-0.030	0.125	-0.001	-0.002	0.000	0.000	-0.002
VIXCLS.l2	0.005	0.138	0.001	-0.006	0.002	-0.001	0.001
T10YIE.l2	4.964	2.969	-0.100	-0.330	0.130	0.230	-0.011
DFF.l2	4.270	-1.186	0.016	-0.054	0.094	0.200	-0.062
DGS1.l2	-0.064	3.946	-0.203	0.525	-0.045	-0.065	-0.018
DGS10.l2	-2.512	-0.232	0.100	-0.110	-0.167	-0.302	0.011
BAA10Y.l2	-3.970	7.004	0.112	-0.112	-0.039	-0.123	-0.033
SPX_rtn.l3	-0.184	0.161	-0.002	0.000	0.003	-0.001	-0.002
VIXCLS.l3	-0.012	-0.019	0.000	0.004	0.001	0.000	-0.002
T10YIE.l3	1.814	-10.380	0.072	0.219	-0.114	-0.161	-0.039
DFF.l3	0.505	-3.608	0.056	0.025	-0.041	-0.015	-0.080
DGS1.l3	6.902	-11.776	0.247	0.035	-0.014	0.247	-0.122
DGS10.l3	-2.654	5.350	-0.059	0.161	0.051	-0.007	0.076
BAA10Y.l3	2.278	4.071	-0.023	0.201	-0.030	0.091	0.018
SPX_rtn.l4	-0.157	-0.151	0.002	-0.006	0.003	0.001	0.000
VIXCLS.l4	0.038	-0.306	0.006	-0.002	-0.001	0.003	0.002
T10YIE.l4	-7.886	10.777	-0.081	-0.027	-0.099	-0.042	0.138
DFF.l4	-3.492	7.691	-0.162	-0.003	-0.056	-0.030	0.150
DGS1.l4	-0.550	-1.386	-0.094	-0.117	-0.025	-0.091	0.094
DGS10.l4	7.101	-4.461	0.028	-0.130	0.155	0.221	-0.178
BAA10Y.l4	13.595	-20.118	0.017	-0.017	0.109	0.061	-0.140
SPX_rtn.l5	0.048	-0.010	0.000	0.002	0.000	0.000	0.000
VIXCLS.l5	0.033	0.170	-0.003	0.002	0.000	-0.002	-0.001
T10YIE.l5	2.101	-1.802	0.041	0.037	0.115	0.121	-0.051
DFF.l5	0.982	-3.487	0.087	0.054	0.069	0.039	-0.002
DGS1.l5	-4.084	6.911	-0.059	-0.128	-0.018	-0.116	0.047
DGS10.l5	-1.648	0.117	0.035	0.114	-0.080	-0.033	0.045
BAA10Y.l5	-9.101	8.867	-0.079	-0.020	-0.054	-0.077	0.048
const (c)	-0.018	0.006	0.001	0.001	-0.001	-0.001	0.002

Figure A.11

VAR STABILITY TEST (JAN 2020 — JUN 2022)

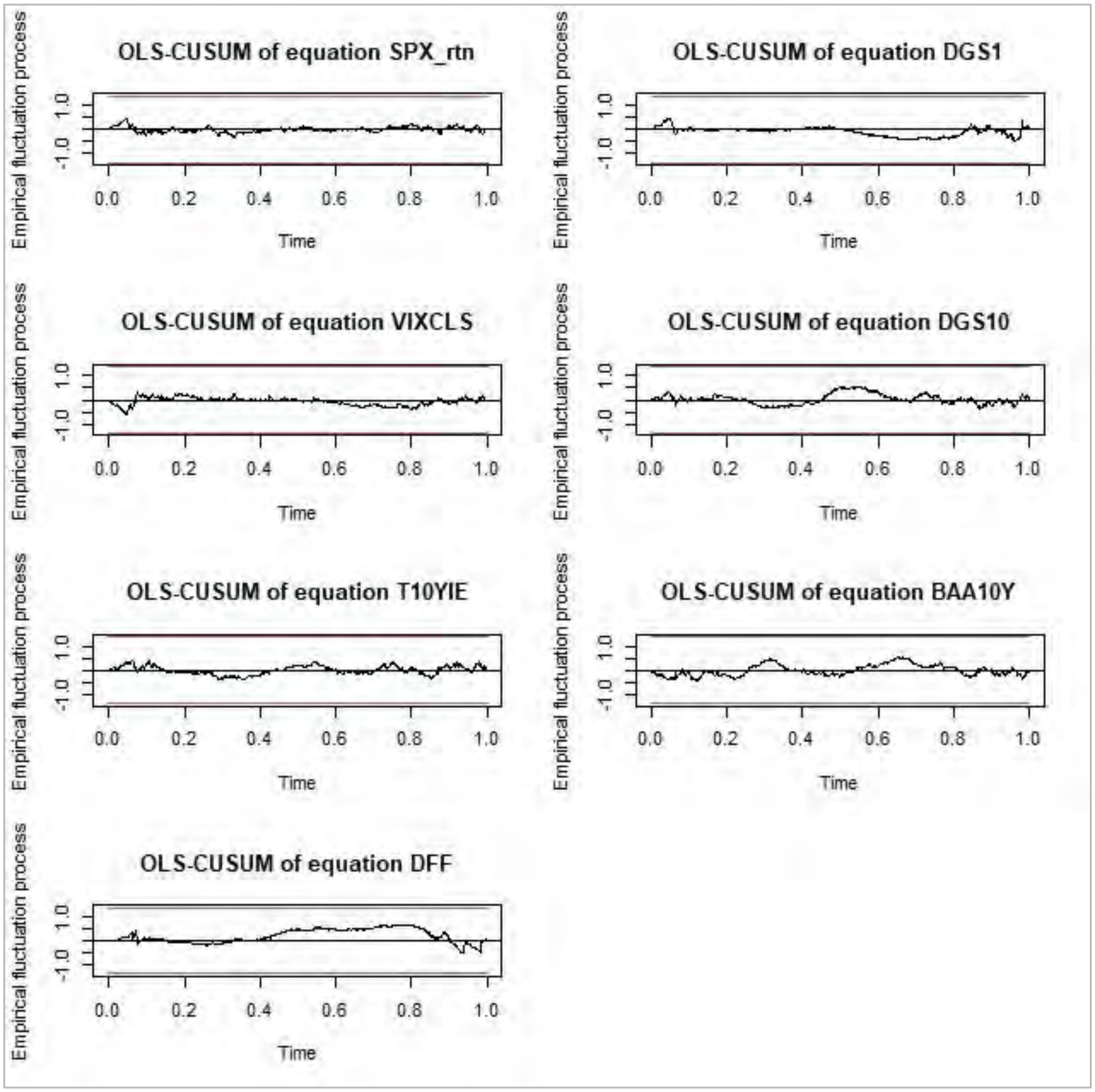


Table A.2

VAR MODEL PARAMETER (JAN 2002 — DEC 2019)

Parameters	SPX_rtn	VIXCLS	T10YIE	DFF	DGS1	DGS10	BAA10Y
SPX_rtn.l1	-0.113	0.044	0.003	0.000	0.000	-0.002	-0.002
VIXCLS.l1	0.028	0.827	0.000	0.000	0.000	0.001	0.000
T10YIE.l1	1.725	-2.175	1.121	-0.003	0.034	-0.016	-0.060
DFF.l1	0.524	-0.458	-0.020	0.791	0.007	0.004	-0.027
DGS1.l1	-1.807	0.326	0.042	-0.033	1.034	-0.033	0.017
DGS10.l1	-0.137	-0.565	-0.051	-0.044	-0.025	1.001	0.020
BAA10Y.l1	-1.155	-3.051	0.046	-0.029	-0.019	-0.096	1.050
SPX_rtn.l2	-0.020	0.049	-0.001	-0.001	-0.002	-0.001	0.000
VIXCLS.l2	-0.035	0.143	-0.001	-0.003	-0.001	-0.001	0.001
T10YIE.l2	-0.812	1.566	-0.163	0.035	-0.009	0.043	0.013
DFF.l2	-0.134	1.016	0.032	-0.030	0.019	-0.001	0.005
DGS1.l2	0.000	1.598	-0.082	0.061	-0.084	-0.029	0.009
DGS10.l2	-0.135	0.012	0.066	-0.029	0.024	-0.051	0.011
BAA10Y.l2	-2.476	6.836	-0.049	-0.020	0.009	0.016	0.087
SPX_rtn.l3	-0.015	-0.032	-0.002	-0.002	0.000	-0.002	-0.002
VIXCLS.l3	-0.008	-0.019	0.000	0.000	0.000	0.000	-0.001
T10YIE.l3	0.217	-0.247	0.028	0.000	-0.049	-0.031	0.023
DFF.l3	0.223	-0.995	-0.007	-0.021	-0.062	0.009	0.013
DGS1.l3	2.728	-1.928	0.064	0.033	0.013	0.084	-0.035
DGS10.l3	-0.773	1.078	-0.005	0.041	0.030	0.046	-0.002
BAA10Y.l3	3.634	-0.282	-0.038	0.063	-0.033	0.018	-0.040
SPX_rtn.l4	-0.060	-0.016	0.000	-0.004	-0.002	-0.001	0.000
VIXCLS.l4	0.020	-0.058	0.002	-0.001	-0.001	0.001	0.001
T10YIE.l4	-0.981	1.707	-0.003	-0.021	0.011	0.007	-0.039
DFF.l4	-0.048	-0.245	-0.003	0.118	0.023	0.006	-0.016
DGS1.l4	-1.428	0.456	-0.058	-0.002	0.071	-0.027	-0.021
DGS10.l4	1.571	-1.289	0.009	-0.053	-0.010	0.018	-0.009
BAA10Y.l4	0.830	-3.360	0.032	-0.099	0.109	0.130	-0.091
SPX_rtn.l5	-0.030	0.032	0.001	0.001	0.000	-0.001	0.000
VIXCLS.l5	0.000	0.080	-0.001	0.003	0.002	-0.001	-0.001
T10YIE.l5	-0.195	-0.816	0.012	-0.004	0.018	0.004	0.061
DFF.l5	-0.496	0.670	-0.001	0.106	0.003	-0.019	0.026
DGS1.l5	0.425	-0.433	0.033	-0.020	-0.024	0.008	0.027
DGS10.l5	-0.541	0.771	-0.019	0.083	-0.020	-0.018	-0.019
BAA10Y.l5	-0.909	0.009	0.008	0.089	-0.066	-0.066	-0.011
const (c)	0.003	0.000	0.000	0.000	0.000	0.000	0.000

Figure A.12
VAR STABILITY TEST (JAN 2002 — DEC 2019)

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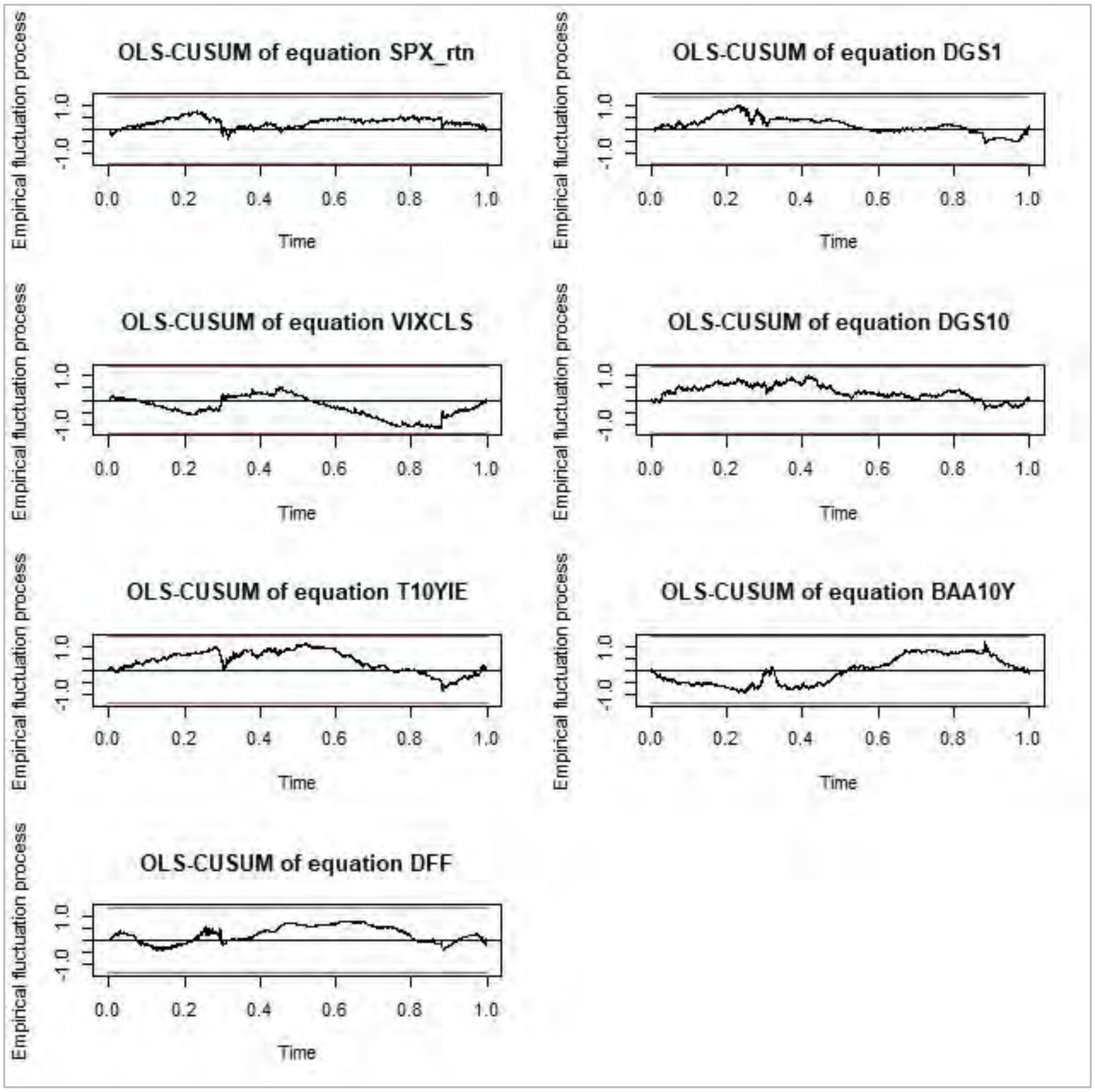


A.5 ASSET ALLOCATON OPTIMIZATION DETAILS

This appendix supplements Section 5.2 Asset Allocation Optimization with more technical details

Economic Scenario Generator

Three economic scenario generators are used for equity return scenario generation: Geometric Brownian motion model, stochastic volatility (SV) model, and stochastic volatility with jumps (SVJ) in returns.

Geometric Brownian motion model

$$\frac{dS_t}{S_{t^-}} = \mu_s dt + \sigma_s dW_t^s$$

Where

S_t: equity index value at time *t.*

W^s: Brownian motion for the equity index.

SV Model

$$\frac{dS_t}{S_{t^-}} = \mu_s dt + \sqrt{\mathrm{V}_{t^-}} dW_t^s$$

$$dV_t = \kappa(\theta - \upsilon \mathrm{V}_{t^-})dt + \sigma\sqrt{\mathrm{V}_{t^-}} dW_t^V$$

Where

V_t: stochastic variance at time t.

W^V: Brownian motion for the stochastic volatility process.

$d\langle W^s, W^v\rangle = \rho_v dt$

SVJ Model

$$\frac{dS_t}{S_{t^-}} = (\mu_s - \bar{\alpha}\Lambda_t)dt + \sqrt{\mathrm{V}_{t^-}} dW_t^s + d\left(\sum_{n=1}^{N_t}\left(e^{Z_n^S} - 1\right)\right)$$

$$dV_t = \kappa(\theta - \upsilon \mathrm{V}_{t^-})dt + \sigma\sqrt{\mathrm{V}_{t^-}} dW_t^V$$

$$d\Lambda_t = \omega dt$$

Where

Λ_t: jump arrival intensity at time t that controls the jump arrival.

N_t: simulated number of jumps based on arrival intensity.

Z_n^S: return jumps that follow $N(\alpha, \delta^2)$.

Both the SV and SVJ models follow the notation convention in Bégin et al. (2021).

Bond fund index follows the Geometric Brownian motion model with its Brownian motion correlated with the equity index Brownian motion, with the correlation coefficient as ρ_S.

$$\frac{dI_t}{I_{t^-}} = \mu_I dt + \sigma_I dW^I$$

Where

I_t: bond fund index value at time *t*.

W^s: Brownian motion for the bond fund index.

$d\langle W^s, W^I \rangle = \rho_s dt$

The parameters used for simulation are set to generate the same level of returns and volatilities in three models for comparison, as shown in Table A.3.

Table A.3
SAMPLE ESG PARAMETERS

Model	Parameters
Geometric Brownian motion (Equity)	μ_s= 0.12 σ_s= 0.19 for baseline and 0.22 for increased volatility
Geometric Brownian motion (Bond)	μ_s= 0.05 σ_s= 0.08
SV	μ_s= 0.12 κ=87.03 θ=0.05 σ=4.16 ρ_v=–0.67
SVJ	μ_s= 0.12 κ=8.34 θ=0.05 σ=1.02 ρ_v=–0.61 ω=2.13 α=-0.01 δ=0.05

Reference dependent utility function in Warren (2019) is used to evaluate each asset allocation plan and rank them. For each asset allocation plan, under each scenario, the utility of the ending funding ratio is calculated and aggregated for each plan using the average value. The best plan has the largest utility score.

$$Utility(FR_t) = \begin{cases} \gamma\left[\left(\frac{FR_t}{TFR}\right)^{\alpha} - 1\right] & if\ FR_t \geq TFR \\ \lambda\left[\left(\frac{FR_t}{TFR}\right)^{\beta} - 1\right] & if\ FR_t < TFR \end{cases}$$

where

FR_t: funded ratio (Asset/Liability) at time t.
TFR: target funded ratio, which is assumed to be 1 in the example.
α: curvature parameter on overfunding, which is assumed to be 0.44.
β: curvature parameter on underfunding which is assumed to be 0.88.
γ: weighting parameter on overfunding, which is assumed to be 1.
λ: weighting parameter on underfunding, which is assumed to be 4.5.

Figure A.13 shows the reference dependent function used to evaluate asset allocation plans.

Figure A.13
SAMPLE UTILITY FUNCTION

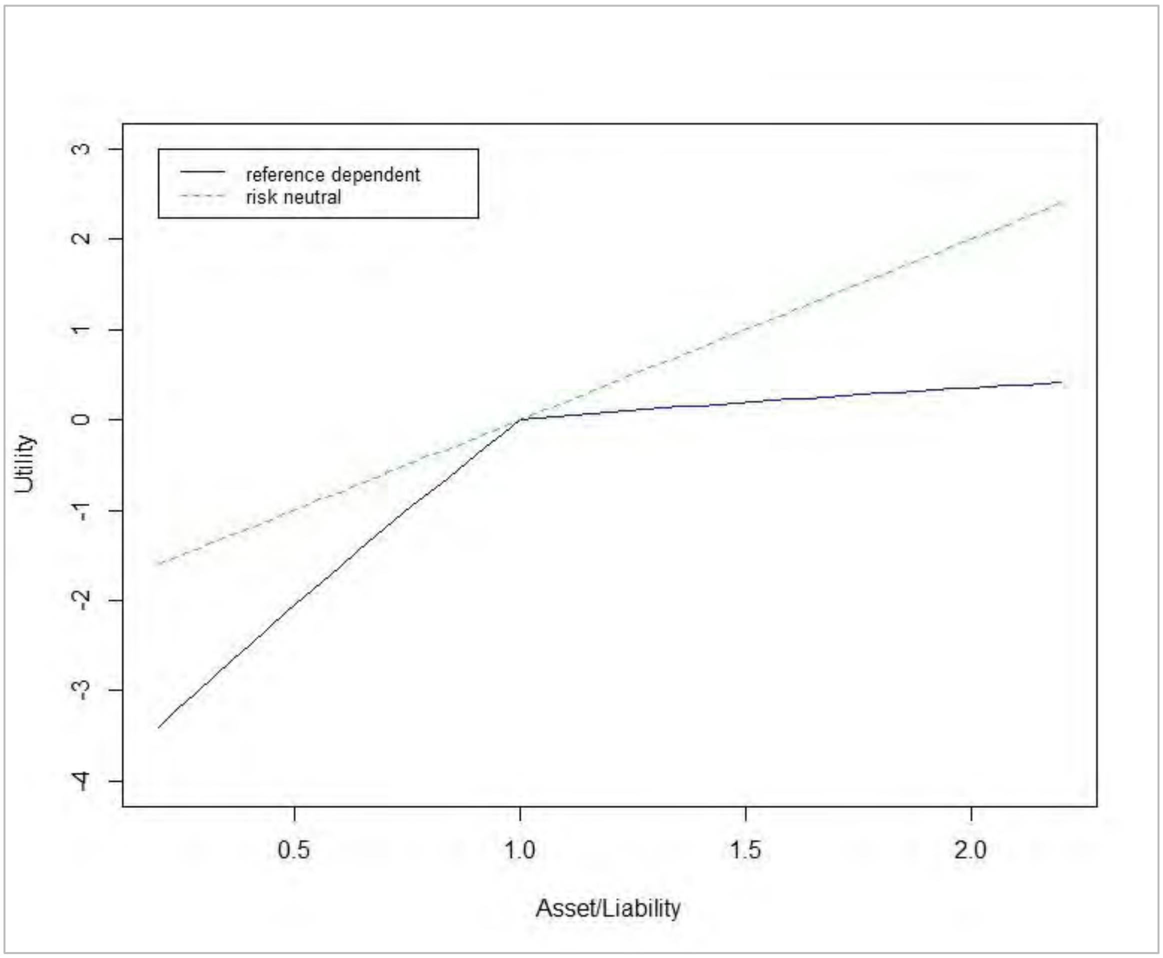


Table A.4 lists the optimal equity allocation plans under different volatility assumption and modeling approaches.

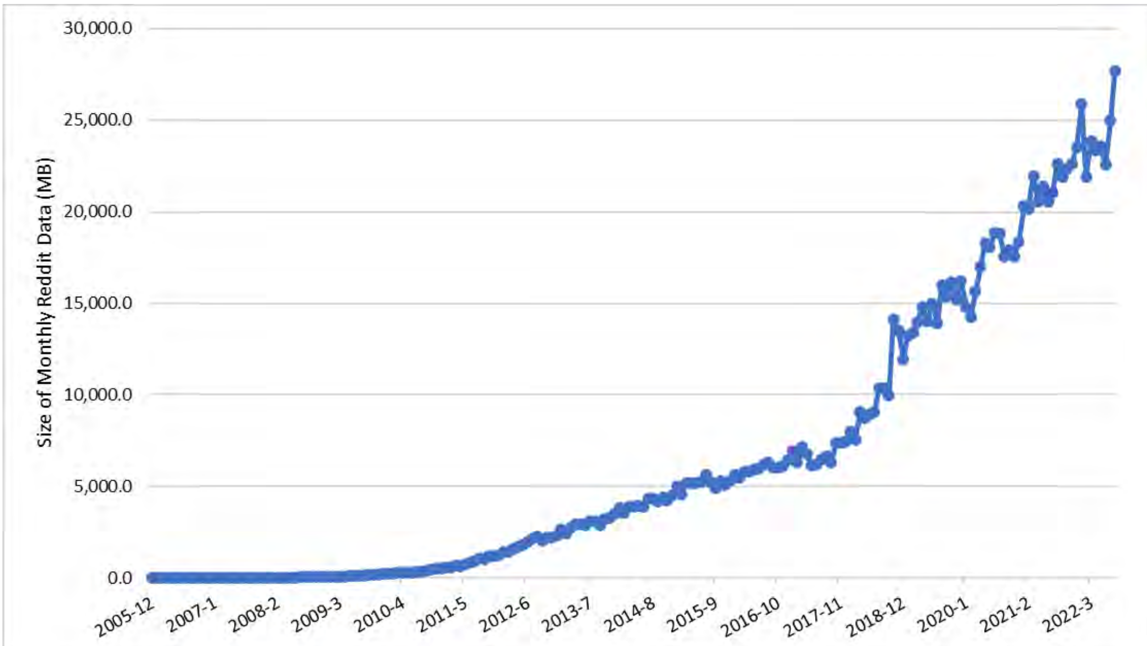
Table A.4
OPTIMAL EQUITY ALLOCATION PLANS

Initial Funding Ratio	Baseline	Increased Vol	Higher Frequency	Stochastic Volatility	Stochastic Volatility with Return Jumps
0.7	1	1	1	1	1
0.725	1	1	1	1	1
0.75	1	1	1	1	1
0.775	1	1	1	1	1
0.8	1	1	1	1	1
0.825	1	1	1	1	1
0.85	1	1	1	1	1
0.875	1	0.56	0.61	0.56	0.53
0.9	0.62	0.38	0.41	0.37	0.37
0.925	0.36	0.26	0.27	0.25	0.25
0.95	0.21	0.16	0.16	0.15	0.15
0.975	0.09	0.07	0.07	0.07	0.07
1	0	0	0	0	0
1.025	0.06	0.04	0.05	0.04	0.04
1.05	0.14	0.1	0.11	0.09	0.08
1.075	0.21	0.15	0.17	0.14	0.13
1.1	0.3	0.21	0.23	0.19	0.18
1.125	0.39	0.27	0.29	0.25	0.24
1.15	0.5	0.34	0.36	0.3	0.29
1.175	0.64	0.4	0.44	0.36	0.35
1.2	1	0.47	0.53	0.44	0.41
1.225	1	0.55	0.65	0.51	0.49
1.25	1	0.65	0.78	0.59	0.58
1.275	1	0.75	1	0.7	0.67
1.3	1	1	1	1	1
1.325	1	1	1	1	1
1.35	1	1	1	1	1
1.375	1	1	1	1	1
1.4	1	1	1	1	1

Appendix B: Reddit Data Analysis

This appendix supplements Section 4.2 to explain detailed text mining methods applied. In total, 1.19TB data is processed covering Reddit comments from December 2005 to June 2022.

Figure B.1
REDDIT DATA SIZE (DECEMBER 2005 — JUNE 2022)



Reddit data is used to understand whether retail investors and social media played a vital role in market volatility. Text mining is based on identifying the most frequently used words in a topic or a specific content class. To facilitate this, words with high frequency but not providing meaningful information may be removed or tagged as unimportant. Words with similar meaning can be aggregated when counting the frequency.

1. All words are converted to lower case so that words with the same characters but in different cases will be counted as the same word.
2. All punctuations are removed.
3. Stop words are removed before the analysis. Stop words are the most frequently used words such as “is”, “at”, “who”, and “that”. They are not useful for extracting key information of the comments.
4. Stemming is the process of removing morphological affixes from words. Words that end with “ed”, “ly”, “es” or “ing” could be trimmed and count as the root of these words. For example, a stemming algorithm may reduce words “snows”, “snowing” and “snowed” to “snow”. This helps to reduce the number of distinct words to be counted across the Reddit comment data.
5. Synonyms are aggregated and counted as the same word. Grouping words with the same meaning as one helps strengthen their value in content classification and leaves more room for other word candidates. However, a word can have different meanings and belong to different grammatical categories in different sentences. For example, the word “good” could mean “nice” as an adjective or “commodity” as a noun. Therefore, syntactic analysis needs to be performed first before determining a word’s synonyms in the context of a sentence. Apache OpenNLP library is used for part-of-speech (POS) tagging. The library uses the maximum entropy model to analyze the

structure of a sentence and provides a category tag to each word of the sentence. Penn English Treebank POS taggers are used as the tagging system. After getting the POS tag for each word in a comment, tags are mapped to word types to facilitate synonym aggregation. Table B.1 lists the tags and the mapping rules used for synonym aggregation.

Table B.1
POS TAG MAPPING

Word Type	Tag
Adjective	JJ Adjective; JJR Adjective, comparative; JJS Adjective, superlative.
Adverb	RB Adverb; RBR Adverb, comparative; RBS Adverb, superlative; WRB Wh-adverb.
Noun	NN Noun, singular or mass; NNS Noun, plural; NNP Proper noun, singular; NNPS Proper noun, plural.
Verb	MD Modal; VB Verb, base form; VBD Verb, past tense; VBG Verb, gerund or present participle; VBN Verb, past participle; VBP Verb, non3rd person singular present; VBZ Verb, 3rd person singular present.

6. With each word in a comment tagged as a word type, WordNet by Princeton University is then used to find out synonyms among these words in the dataset. WordNet is a large lexical database of English words (adjectives, adverbs, nouns and verbs) organized as groups of synonyms. Synonyms are then aggregated. Only one word will be used to represent a group of synonyms after the aggregation.

Another possible adjustment is spell check and correction. A majority of the comments are written by using mobile phones and tablets that have spell check. In addition, the accuracy of automatic spelling correction is not very satisfactory. Therefore, spell check is not conducted for this research.

The most frequent 200 words are used with their daily frequency counted and used as explanatory variables. Bigrams (two-word combinations) and trigrams (three-word combinations) are also tested without improvement in the regression model performance.

In addition to word frequency, sentiment of Reddit comments is also assessed and used for the attribution analysis. Each Reddit comment is assigned with a sentiment score, with the daily sentiment calculated using either simple arithmetic averaging or weighted averaging with Reddit scores. Pre-trained sentiment analyzers are tested including Valence Aware Dictionary and Sentiment Reasoner (VADSR) which is pre-trained and suited for analyzing social media data, and Stanford CoreNLP. VADSR is used given its better performance. It provides the scores that a comment is positive, neutral, or negative. It also gives a compound score which is calculated using the scores of the three categories. The compound score is used to assess the aggregate sentiment of each Reddit comment.

Appendix C: Feature Importance

Feature importance is a standard step in model validation. It is beneficial in three ways.

- If some unexpected variables show up in the list of important features, it helps identify potential issues with the model and data and requires further investigation before implementing the model.
- Important features can be used to set up key risk indicators and be frequently monitored for material changes.
- In the presence of overfitting, unimportant features may be removed.

The method of determining feature importance varies by model.

Linear regression including Lasso and Ridge regression: The explanatory variables are normalized to the same range before model fitting. A variable's importance is measured by the absolute value of the coefficient of that variable.

CART: A variable's importance is measured by the increase of data purity because of a split based on that variable. For a regression problem, the importance of variable x_i can be calculated as follows.

$$Imp_{regression}(x_i) = \frac{1}{T}\sum_{t=1}^{T}\left(\sum_{s=1}^{S}\frac{N_L \cdot N_R}{N_L + N_R}(\overline{Y_L} - \overline{Y_R})^2 \cdot Ind(split = x_i)\right)$$

Where

x_i: the *i*th input variable.

T: total number of CART models in the RF model.

S: total number of splits in a CART model.

$\overline{Y_L}$: the mean of Y in the left node after the split.

$\overline{Y_R}$: the mean of Y in the right node after the split.

N_L: the number of records in the left node after the split.

N_R: the number of records in the right node after the split.

$Ind(split = x_i)$: indicator function with a value of 1 if the split is based on variable x_i and a value of 0 otherwise.

For a classification problem, the measure $\frac{N_L \cdot N_R}{N_L+N_R}(\overline{Y_L} - \overline{Y_R})^2$ needs to be replaced. A possible measure is the improvement of the Gini impurity index $G(N)$, as defined below.

The Gini index is commonly used to represent the data dispersion. It is calculated as follows.

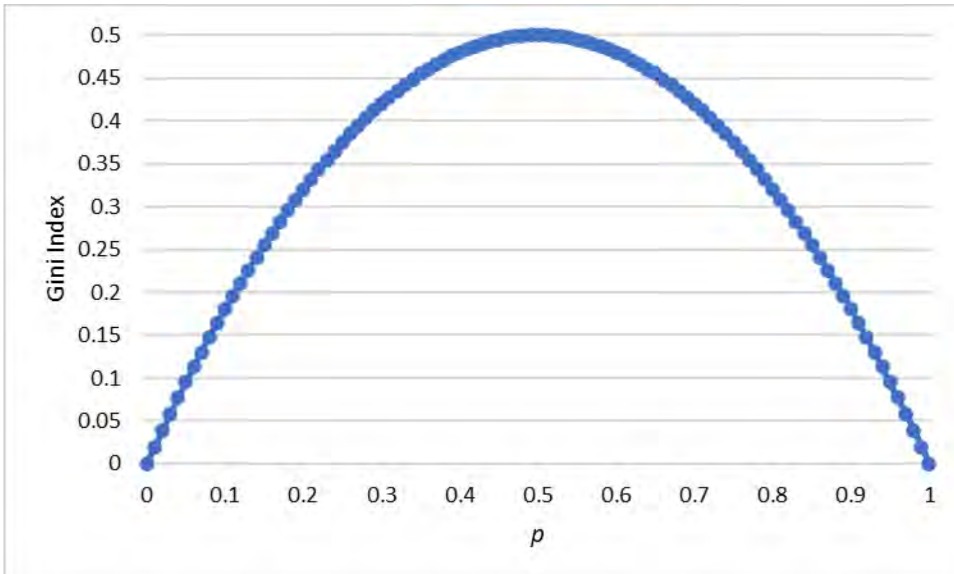
$G(T) = \sum_{i=1}^{n} p_i(1 - p_i) = 1 - \sum_{i=1}^{n} p_i^2$

Where

p_i: the probability that the data belongs to category i.
n: the number of categories in the data.
T: the dataset based on which Gini index is calculated.

If the data is pure, meaning that it only has one value, the Gini index is zero. If the data is evenly dispersed, such as 50% probability for each of two possible values, the Gini index is 0.5. Figure C.1 shows the Gini index curve for data with only two categories. The Gini index reaches the maximum when the probabilities are even between two categories.

Figure C.1
GINI INDEX CURVE



At each split, the increase in data purity in subsets is maximized when choosing the variable and the threshold for splitting.

$$\max_{x,threshold} G(T) - p(T_L)G(T_L) - p(T_R)G(T_R)$$

Where

T_L: the data subgroup of the split's left branch.
T_R: the data subgroup of the split's right branch.
p: the portion of the data subgroup in the dataset before splitting.
x: the variable to be used for the splitting.
$threshold$: the threshold used to set the split based on the value of x.

Assuming that the data is evenly dispersed with 50% probability for each of the two categories, the Gini index $G(T)$ before splitting is 0.5. If the split divides the data perfectly into the two categories, then the new Gini index is zero, as calculated below. The gain from the split is 0.5 at its maximum.

$$p(T_L)G(T_L) + p(T_R)G(T_R) = 0.5 \times 0 + 0.5 \times 0 = 0$$

For each split based on variable x_i, the Gini importance can be measured as the reduction in the Gini index:

$$Gini\ Imp(x_i) = (N_L + N_R)G(N) - N_L G(N_L) - N_R G(N_R)$$

If the variable is used in multiple splits, the Gini importance is aggregated for the variable:

$$Imp_{classification}(x_i) = \frac{1}{T}\sum_{t=1}^{T}\left(\sum_{s=1}^{S} Gini\ Imp(x_i) \cdot Ind(split = x_i)\right)$$

Random Forests: A variable's importance can be measured as the average importance level in each individual CART in the Random Forests model.

GBM: A variable's importance can be measured as the total importance level in each individual CART in the sequential tree models.

Appendix D: Open-Source R Program

R codes are created for education purposes and hosted at https://github.com/Society-of-actuaries-research-institute/FP104-Market-Volatility-Risk-in-an-Era-of-Extreme-Events.

The program is self-explained with input data available in the same GitHub repository.

Input data files:

- "daily_data.csv." It contains all the daily data used for studying historical market volatility and prediction modeling in attribution analysis.
- "monthly_data.csv." It contains all the monthly data used for studying historical market volatility.

R script: "mkt_vol_program.r." It contains the codes used to generate results presented in the following:

- Section 2: Historical Market Volatility Behavior
- Section 3: Attribution Analysis
- Section 4.2: Asset Allocation Optimization
- Appendix A: Market Volatility Analysis

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